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A Curvature Issue in the Optimal Commodity Tax Problem

Arielle Adiri (Undergraduate Research Fellow)
and Prof. Jesse A. Schwartz

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Arielle Adiri* and Jesse A. Schwartz
aadiri@students.kennesaw.edu, jschwar7@kennesaw.edu

Department of Economics, Finance and Quantitative Analysis, Kennesaw State University

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Abstract

In the optimal commodity tax problem, we show that the Kuhn-Tucker conditions are neither necessary nor sufficient for optimization. The inadequacy stems from a curious economic insight: indirect utility functions are quasiconvex in prices. Less formally, for any two lopsided price vectors with some prices high and some low such that the consumer fares equally well, the averaged price vector leaves the consumer worse off. With lopsided prices, the consumer can splurge on the less expensive goods and conserve on the more expensive goods. This insight is at odds with the conventional wisdom of splitting taxes across goods to equalize marginal deadweight losses, supported by the first-order conditions from the optimal tax problem. We show with an admittedly contrived example that the Ramsey tax formulas derived from such first-order conditions can actually *minimize* consumer welfare due to the general quasiconvexity of indirect utility (always) and linearity of the tax revenue constraint (our example). Though contrived, the consumer's preferences in our example satisfy standard assumptions. Our work suggests a rethinking of optimal taxes is called for to better accommodate a consumer's preference for lopsided prices.

Keywords: commodity taxes, quasiconvex indirect utility functions, publicly provided private goods, Ramsey taxes, Stone-Geary utility

JEL Codes: B210 (History of Economic Thought: Microeconomics), H21 (Optimal Taxation), H42 (Publicly Provided Private Goods)

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1 Introduction

"I like to pay taxes" is a sentence that you do not hear too often. Rather, people like to avoid taxes. But if many goods are taxed it is difficult to avoid taxes. Alternatively, if some goods have high taxes and others low taxes and if consumers CAN effectively avoid taxes, meeting a tax revenue requirement may be thwarted. Still, if consumers like to avoid taxes, this boon should be kept in mind when designing tax policy.

How should taxes be spread across goods? This is a famous problem in economics. Conventional wisdom says to tax goods more if inelastically demanded or supplied, but this wisdom derives either from Ramsey tax formulas (more later) or from partial equilibrium analysis only justified under special assumptions about the independence of goods and demand not having income effects.

We will give an example where the optimal taxes are not spread out evenly contradicting what the usual (Ramsey) techniques call for. Instead, the optimal taxes call for taxing only one of the two goods eligible for taxation. (If all goods could be taxed, it is optimal to tax them all by an equal percentage. The Ramsey tax problem is a second-best problem.)

2 Notation

Let us specify our notational conventions. Bold faced lowercase letters indicate vectors and nonbold faced letters indicate one-dimensional variables. This gives $\mathbf{x} = (x_1, \ldots, x_n)$ as a consumption bundle with x_i giving the quantity of good i. And $\mathbf{p} = (p_1, \ldots, p_n)$ is the vector of prices with p_i the price of good i. A consumer's income will be denoted as m (for money). We will use uppercase letters to denote functions. For example, $U(\mathbf{x})$ is a utility function, or when we need to define some mathematical property of a function we will use the generic F. We will use $V(\mathbf{p}, m)$ for the indirect (maximized) utility function. We will use capital letters near the beginning of the alphabet to indicate particular values of some variable, so that $\mathbf{p}_A$ and m_A might denote the price vector and income in scenario A, while x^*_{1A} would be the optimal quantity of good 1 in this scenario.

3 Overview of the Ramsey Tax Problem

To focus on the issue at hand—the wrong curvature of the objective function—we consider the most basic of optimal commodity tax problems, as treated for example in Varian

(1992) or Dixit (1970). These treatments are variations of Frank Ramsey's groundbreaking (1927) paper.[1] The essential idea of Ramsey, and the part that we concentrate on, is that the social planner chooses commodity taxes to maximize a representative individual's maximized utility from her choice of goods, the individual taking the prices including taxes as fixed. That is, there is a maximization problem within a maximization problem. Some treatments include production; we and Varian (1992) consider only consumption and assume prices without taxes are fixed (as would be the case if the country was small and engaged in free trade). As Runge (2020) summarizes, Ramsey's result says that efficient taxation sets "the tax for each commodity in relation to price elasticities of supply and demand after accounting for the income effect, rather than imposing a single tax rate on all [taxable] goods. Although each of these calibrated taxes creates a small distortion, their granularity avoids the larger distortion of a blunt single tax." We should note that Ramsey (1927) justified his approach for either case of the revenue requirement being infinitesimally small (in which case the tax problem is not so interesting, right?) or the case in which utility function is pairwise quadratic with terms involving x_i and x_j and $x_i x_j$ for each pair goods i and j. Future treatments ala Varian (1992) and Dixit (1970) assume Ramsey's general approach without such limitations.

In a nutshell, there is a social planner and a representative consumer. The representative consumer chooses her consumption bundle to maximize her utility subject to her budget constraint, taking prices as given. The social planner will choose taxes and thereby affect the after-tax prices the consumer faces.

Let us begin with the consumer. Because we will later argue that our example of the consumer's preferences are quite commonplace, we will back up all the way to the consumer's preferences. A consumption bundle is a vector of n goods in $\mathbf{R}_+^n$. The consumer's preference relation $\succsim$ can be used in expressions like $\mathbf{x}_A$ and $\mathbf{x}_B$ to indicate that the consumer likes $\mathbf{x}_A$ at least as much as $\mathbf{x}_B$. Axioms we will always invoke are as given below.

(Complete): For all $\mathbf{x}_A, \mathbf{x}_B \in \mathbb{R}_+^n$, either $\mathbf{x}_A \succsim \mathbf{x}_B$ or $\mathbf{x}_B \succsim \mathbf{x}_A$ (or both).

Aside: As is typical, we use $\sim$ for indifference and $\mathbf{x}_A \sim \mathbf{x}_B$ if $\mathbf{x}_A \succsim \mathbf{x}_B$ and $\mathbf{x}_B \succsim \mathbf{x}_A$; we use $\succ$ for strict preference and $\mathbf{x}_A \succ \mathbf{x}_B$ if $\mathbf{x}_A \succsim \mathbf{x}_B$ but not $\mathbf{x}_B \succsim \mathbf{x}_A$.

(Reflexive): For all $\mathbf{x}_A \in \mathbb{R}_+^n$, we have $\mathbf{x}_A \succsim \mathbf{x}_A$.

[1] A nice discussion of Ramsey's three important contributions, including optimal taxation, is given in Runge (2020) This article also contains some interesting details about his tragically short life of 27 years, including that his intellect could be compared with Alan Turing, John von Neumann, David Hilbert, and Ludwig Wittgenstein.

(Transitive): For all $\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C \in \mathbb{R}_+^n$, we have $\mathbf{x}_A \succsim \mathbf{x}_B$ and $\mathbf{x}_B \succsim \mathbf{x}_C$ implies $\mathbf{x}_A \succsim \mathbf{x}_C$.

(Continuous): For any $\mathbf{x}_A \in \mathbb{R}_+^n$, the better than set $B(\mathbf{X} = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{x} \succsim \mathbf{x}_A\}$ and the worse than set $W(\mathbf{X} = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{x}_A \succsim \mathbf{x}\}$ are closed sets.

Under the above assumptions, such preferences can be represented by a continuous utility function $U : \mathbb{R}_+^n \to \mathbb{R}$ such that $U(\mathbf{x})$ gives the consumer's utility (score) for consumption bundle $\mathbf{x}$, such that a $U(\mathbf{x}_A) \geq U(\mathbf{x}_B)$ if and only if $\mathbf{x}_A \succsim \mathbf{x}_B$.[2] We will sometimes refer to additional assumptions.

(Strongly Monotonic): If $\mathbf{x}_A, \mathbf{x}_B \in \mathbb{R}_+^n$ satisfy $x_{iA} \geq x_{iB}$ for $i = 1, 2, \ldots, n$ with at least one of these inequalities strict, then $\mathbf{x}_A \succ \mathbf{x}_B$.

(Weakly Monotonic): If $\mathbf{x}_A, \mathbf{x}_B \in \mathbb{R}_+^n$ satisfy $x_{iA} > x_{iB}$ for $i = 1, 2, \ldots, n$, then $\mathbf{x}_A \succ \mathbf{x}_B$.

Note that strictly monotonic preferences are always weakly monotonic, but not vice versa.

(Weakly Convex): For any $\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C \in \mathbb{R}_+^n$ with both $\mathbf{x}_A \succsim \mathbf{x}_C$ and $\mathbf{x}_B \succsim \mathbf{x}_C$, we have $t\mathbf{x}_A + (1-t)\mathbf{x}_B \succsim \mathbf{x}_C$ for all $t \in [0, 1]$.

(Strictly Convex): For any $\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C \in \mathbb{R}_+^n$ with $\mathbf{x}_A \neq \mathbf{x}_B$ and with both $\mathbf{x}_A \succsim \mathbf{x}_C$ and $\mathbf{x}_B \succsim \mathbf{x}_C$, we have $t\mathbf{x}_A + (1-t)\mathbf{x}_B \succ \mathbf{x}_C$ for all $t \in (0, 1)$.

Strictly convex preferences give C-shaped indifference curves; weak convexity allows for linear portions along an indifference curves. All strictly convex preferences are weakly convex, but not vice versa.

We note that Cobb-Douglas preferences given by $U(x_1, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where the powers are positive and sum to 1 are merely weakly monotonic and weakly convex, easiest to see when $n = 2$. In this case, the x_1-axis and x_2-axis form the $U = 0$ indifference curve and so along the x_1-axis the consumer getting more good 1 while holding x_2 at 0 maintains utility of 0 and such an indifference curve has two flat portions—the x_1-axis and the x_2-axis. We note this to emphasize that our main example in section 6 will be Stone-Geary which satisfies the same properties as the canonical Cobb-Douglas.

The representative consumer with income m and utility function $U(\mathbf{x})$, solves

[2]See Mas-Colell et al. (1995) Proposition 3.C.1 for the proof.

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} \; U(\mathbf{x}) \text{ st } \mathbf{p} \cdot \mathbf{x} = m.$$

Let $\mathbf{x}(\mathbf{p}, m)$ be the vector of demand functions.[3] The *indirect* utility function is then $V(\mathbf{p}, m) = U\Big(\mathbf{x}(\mathbf{p}, m)\Big)$. Let λ denote the multiplier used for the budget constraint when setting up the Lagrangian for the consumer's utility maximization problem:

$$L = U(\mathbf{x}) - \lambda\Big[\mathbf{p} \cdot \mathbf{x} - m\Big].$$

Ramsey and more modern treatments call upon Roy's Identity (an application of an envelope theorem) to get

$$\frac{\partial V(\mathbf{p}, m)}{\partial p_i} = -\lambda x_i(\mathbf{p}, m). \tag{1}$$

Aside: λ like the optimal choice of $\mathbf{x}$ in the consumer's utility maximization problem will depend on prices and income, but nevertheless that dependence is usually suppressed as in equation (1).

Now we turn to the social planner's problem. In the simplest version of the problem that we take up, the government has an exogenous revenue requirement of R dollars and the economy is a small and open such that there is a fixed price vector $\mathbf{p} = (p_1, \ldots, p_n)$. There is a tax vector $\mathbf{t} = (t_1, \ldots, t_k, 0, \ldots, 0)$, where the first $k < n$ goods can have a tax but goods $k+1$ on are ineligible for taxation. These could be blackmarket goods, for example, or goods where the transactions costs of collecting the tax is prohibitive (goods sold at flea markets), or goods where it would be politically unpopular to tax them (insulin or consumer staples like bread and milk). Overall, the consumer will face after-tax prices of $\mathbf{p} + \mathbf{t}$. We preclude taxation of all goods because in that special case, the optimum is to tax all goods by the same percentage sufficiently to meet the revenue requirement (see Dixit, 1970).

The social planner then assumes the representative consumer takes the prices with taxes as given and maximizes utility. The social planner maximizes the maximized utility of this representative subject to meeting the exogenous revevue constraint:

$$\max_{\mathbf{t} \in \mathbb{R}^n_+} \; V(\mathbf{p} + \mathbf{t}, m) \text{ st } \sum_{i=1}^{k} t_i x_i(\mathbf{p} + \mathbf{t}, m) = R$$

[3]See Milgrom and Segal (2002) which alternatively gives a "selection" if there are multiple solutions to the optimization problem. The other potential issue is that no solution exists; this is not an issue given we have a continuous utility function and closed and bounded budget set.

with corresponding Lagrangian:

$$\mathcal{L} = V(\mathbf{p}+\mathbf{t}, m) - \mu\left[\sum_{i=1}^{k} t_i x_i(\mathbf{p}+\mathbf{t}, m) - R\right].$$

Now it is standard that any saddlepoint of $\mathcal{L}$ will involve the optimal choice of taxes, but the question we address is if that saddlepoint can be found using the Kuhn-Tucker conditions. We will next argue that the usual curvature on the "indifference curves" is fundamentally incorrect; we set off indifference curves to note that in the social planner's problem we will use indifference curves of the indirect utility function over the $\mathbf{p}+\mathbf{t}$ vectors rather than over consumption bundles when using direct utility functions. To preview our results, we build an example where the tax revenue constraint is linear, but the indifference curves for indirect utility have the wrong curvature, so that using the first-order conditions for Kuhn-Tucker does not find the saddlepoint and indeed guarantees a minimum rather than a maximum in our example. Before doing that, we next explain the shape of these indifference curves.

4 Quasiconvexity of Indirect Utility in Prices

With preferences over goods, it is common that averages are often preferred to extremes. For example, a person would want a mix of jazz and rock albums, a mix of business and casual clothing, or a mix of chicken and steak. It is natural to think that a person would prefer variety over a multitude of products. However, another valid axiom is monotonicity of preferences, people always want more: more albums, more clothes, and more meat is always preferred to less. The lopsided prices allow the consumer to buy more stuff, even though the stuff may be less diverse. It is an example of a different kind of trade-off than chicken and steak, consuming more stuff makes up for the now lopsided consumption bundle, or in other words, the lack of variety.

	p_1	p_2	x_1^*	x_2^*	U
Scenario A	20.00	5.00	2.5	10	5
Scenario B	5.00	20.00	10	2.5	5
Scenario C	12.50	12.50	4	4	4

Table 1: Extreme Price Vectors Preferred to Averages

Consider a consumer with Cobb-Douglas preferences as represented by utility function $U(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. It is well-known (and straightforward to derive) that the solution to such a consumer's utility maximization problem is always for her to spend half of her money on good 1 and half on good 2; this example will expedite our illustration that consumers prefer lopsided prices over even prices. Consider scenario A when this consumer has money $m = \$100$ to spend when prices are $p_{1A} = \$5$ and $p_{2A} = \$20$. Thus, she does best to spend \$50 on each good or she optimally consumes $x_{1A}^{\star} = 10$ and $x_{2A}^{\star} = 2.5$. The indirect (maximized) utility function is

$$V(p_1, p_2, m) = \left[\frac{m}{2p_1}\right]^{1/2} \left[\frac{m}{2p_2}\right]^{1/2} = \frac{m}{2(p_1 p_2)^{1/2}}$$

where number in brackets are the demand functions for goods 1 and 2.[4]

Thus, in scenario A, maximized utility is $V_A = V(5, 20, 100) = 10^{1/2} 2.5^{1/2} = 5$. Alternatively, consider scenario B with the lopsided prices flip-flopped: $p_{1B} = \$20$ and $p_{2B} = \$5$, but holding fixed income at $m = \$100$. In this case she does best buying 2.5 units of good 1 and 10 units of good 2 and obtains indirect utility of $V_B = 5$ again.

Next, consider scenario C, where the consumer still has $m = \$100$, but with prices averaged over scenarios A and B: $p_{1C} = \$12.50$ and $p_{2C} = \$12.50$. In scenario C she does best to buy 4 units each of goods 1 and 2 and obtains indirect utility of only 4 utils. This means that the consumer likes the original lopsided prices of scenarios A or B better than the evened out prices of scenario C.

This example helps illustrate why this result might be true, especially if we consider particular pairs of goods. They could be pounds of chicken and pounds of steak; or work outfits and leisure outfits; or rock and jazz albums. These are quotidian examples. There is a splurge vs. conserve story. Although we have illustrated this story with examples of goods that are substitutes and Cobb-Douglas preferences for expediency, this result is very general and comes about purely because of the consumer's optimization.

We next formalize and generalize this example to provide a curious but standard result in microeconomics. We now rehash some usual structure in microeconomics.

Definition: The function $F : X \to \mathbb{R}$, where X is some convex set in $\mathbb{R}^n$ is **quasiconvex** if

[4] We thank Tim Mathews for encouraging expressing V emphasizing the product of prices in the denominator, which drives the quasiconvexity of V in this example.

for all $\lambda \in (0, 1)$ and all $\mathbf{x}_A, \mathbf{x}_B \in X$ with $F(\mathbf{x}_A) \leq F(\mathbf{x}_B)$, we have

$$F(\lambda \mathbf{x}_A + [1 - \lambda]\mathbf{x}_B) \leq F(\mathbf{x}_B).$$

It is immediate that if F is quasiconvex and $F(\mathbf{x}_A) = F(\mathbf{x}_B)$, then $F(\lambda \mathbf{x}_A + [1 - \lambda]\mathbf{x}_B) \leq F(\mathbf{x}_A)$. An equivalent way of thinking about quasiconvex functions is that worse-than sets are convex sets, summarized by the following lemma.

Lemma 1. *Consider function $F : X \to \mathbb{R}$, where X is some convex set in $\mathbb{R}^n$. Then F is quasiconvex if and only if for all values $\alpha \in \mathbb{R}$ and all $\mathbf{x}_A, \mathbf{x}_B \in X$ with $F(\mathbf{x}_A), F(\mathbf{x}_B) \leq \alpha$ and all $\lambda \in (0, 1)$, we have $F(\lambda \mathbf{x}_A + [1 - \lambda]\mathbf{x}_B) \leq \alpha$.*

Remarks: $\lambda \mathbf{x}_A + [1 - \lambda]\mathbf{x}_B$ with $0 < \lambda < 1$ is the *convex combination* of these two vectors and by definition a set X is *convex* if the convex combination of any two points in the set is also contained in the set; a worse-than set given some function $F : X \to \mathbb{R}$ and some constant α can be described as any $\mathbf{x} \in X$ with $F(\mathbf{x}) \leq \alpha$.

We next give the standard result that gives the curvature of indirect utility functions in prices and include the proof—though standard—since so much of our paper depends on this result.

Lemma 2. *The indirect utility function $V(\mathbf{p}, m)$ is quasiconvex in the price vector* $\mathbf{p}$.

Proof. Select any $\lambda \in (0, 1)$ and any $\mathbf{p}_A, \mathbf{p}_B \in \mathbb{R}^n_+$ such that $V(\mathbf{p}_A, m) \leq V(\mathbf{p}_B, m)$. Form the average price vector $\lambda \mathbf{p}_A + (1 - \lambda)\mathbf{p}_B$. The resulting budget sets are:

$$\begin{aligned}
BS_A &= \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}_A \cdot \mathbf{x} \leq m\} \\
BS_B &= \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}_B \cdot \mathbf{x} \leq m\} \\
BS_C &= \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}_C \cdot \mathbf{x} \leq m\}.
\end{aligned}$$

We next establish that $BS_C \subset BS_A \cup BS_B$; or equivalently: if $\mathbf{x} \in BS_C$, then $\mathbf{x} \in BS_A$ or $\mathbf{x} \in BS_B$; or equivalently, the contrapositive: If $\mathbf{x} \notin BS_A$ and $\mathbf{x} \notin BS_B$, then $\mathbf{x} \notin BS_C$. To see this last holds, select $x \in \mathbb{R}^n_+$ such that

$$\begin{aligned}
&\mathbf{p}_A \cdot \mathbf{x} > m \text{ or } \lambda \mathbf{p}_A \cdot \mathbf{x} > \lambda m \\
&\mathbf{p_B} \cdot \mathbf{x} > m \text{ or } (1 - \lambda)\mathbf{p}_B \cdot \mathbf{x} > (1 - \lambda)m.
\end{aligned}$$

Summing then gives $\mathbf{p}_C \cdot \mathbf{x} > m$ so that $\mathbf{x} \notin BS_C$, thereby establishing the contrapositive.

Let $\mathbf{x}_A, \mathbf{x}_B$, and $\mathbf{x}_C$ denote the optimal choices under the three scenarios, so that indirect utility is $V(\mathbf{p}_A, m) = U(\mathbf{x}_A)$ and $V(\mathbf{p}_B, m) = U(\mathbf{x}_B)$ and $V(\mathbf{p}_C, m) = U(\mathbf{x}_C)$. Because $BS_C \subset BS_A \cup BS_B$, we have:

$$\mathbf{x}_C \in BS_A \implies V(\mathbf{p}_C, m) \leq V(\mathbf{p}_A, m) \leq V(\mathbf{p}_B, m) \text{ or}$$
$$\mathbf{x}_C \in BS_B \implies V(\mathbf{p}_C, m) \leq V(\mathbf{p}_B, m).$$

□

5 Geometry of Optimal Tax Problem

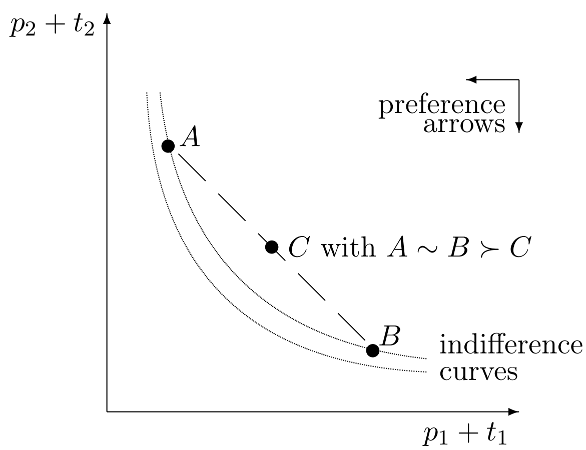

Figure 1: Quasiconvex Indirect Utility

The punchline of our story is that indirect utility is quasiconvex in prices (or prices plus taxes) and so indifference curves look like as illustrated in Figure 1. The Cobb-Douglas example in the previous section gave a numerical illustration of what the geometry shows. Thus, understanding that a convex combination of two price vectors on the same indifference curve is in the worse than set gives the usual C-shaped indifference curves. But because prices are bads (consumers like lower prices), the consumer does better on indifference curves closer to the origin.

Now what we do not know is the shape of the revenue requirement line

$$\sum_{i=1}^{k} t_i x_i(\mathbf{p} + \mathbf{t}, m) = R. \tag{2}$$

This generally depends on the demand functions. What we do in the next section is to greatly simplify matters by carefully selecting the utility function so that resulting demand functions make equation (2) linear. This will then mean that solving for the tangency gives us a minimum amongst all points on the revenue requirement line, as illustrated below.

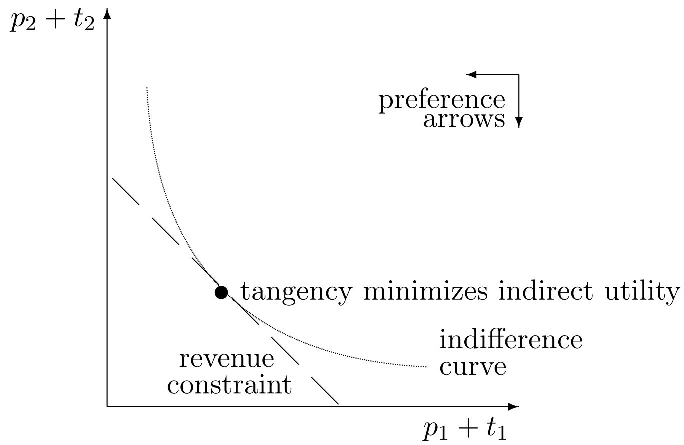

Figure 2: Suboptimal Taxes

6 Stone-Geary Example

6.1 Stone-Geary Utility Max and Demand Functions

Our main example to show the limitations of the Ramsey tax approach is as follows. Suppose that there are three goods, and only goods 1 and 2 are eligible for taxes. Suppose for now that goods 1 and 2 are publicly provided private or club goods (i.e., any goods that are excludable). For instance, the goods could be rides on a ferry, rides on a subway, or some kind entry price to a national park. It could also be for some kind of government stamp on a passport. (Our examples are attempts at goods with low marginal cost so that we may abstract away from such concerns and concentrate on revenue, for now.) Good 3 can be privately provided but it is not taxable for any number of reasons: transaction costs of collecting is too high, enforcing the tax is too costly, or any "under the table" payments like for babysitting.

We usually understand with goods that averages are preferred to extremes, as formalized with the assumption of strict convexity as given in section 3. A mixture of steak and chicken during the month is preferred to chicken every night. However, another valid axiom is monotonicity of preferences: a consumer is better off with more albums, more clothes, more meat. What the lopsided prices do is allow the consumer to buy more stuff even though she ends up with a more lopsided consumption bundle. But just as with any two goals (here more consumption and averaged consumption), the consumer is willing to tradeoff between goals: the more consumption allowed by lopsided prices more than makes up for the resulting uneven consumption bundle, as we showed with the Cobb-Douglas example in the section 4 and more formally by the quasiconvexity of indirect utility in Lemma 2.

A Stone-Geary function over three goods $U : \mathbb{R}^3_+ \to \mathbb{R}$ is defined as follows:

$$U(x_1, x_2, x_3) = \begin{cases} (x_1 - \bar{x}_1)^a (x_2 - \bar{x}_2)^b (x_3 - \bar{x}_3)^c & \text{if } x_1 \geq \bar{x}_1 \text{ and } x_2 \geq \bar{x}_2 \text{ and } x_3 \geq \bar{x}_3 \\ \min\{x_1 - \bar{x}_1, x_2 - \bar{x}_2, x_3 - \bar{x}\} & \text{otherwise.} \end{cases}$$

where $a, b, c \geq 0$, $a + b + c = 1$ and $\bar{x}_1, \bar{x}_2, \bar{x}_3 \geq 0$. The Cobb-Douglas utility functions are special cases of the Stone-Geary utility functions in which the necessary requirements $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are all set to zero. Typically, the Stone-Geary function is only defined when consumption exceeds the necessary requirements $\bar{x}_1, \bar{x}_2, \bar{x}_3$ with the otherwise part set equal to zero, but this would violate weak monotonicity. Thus, we have included the minimum and this makes the given utility function satisfy preferences that are weakly monotonic and weakly convex, as well as the other assumptions of complete, reflexive, transitive, continuous. Indeed, for consumptions bundles $(x_1, x_2, x_3) \gg (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $\gg$ means each term is strictly greater, preferences will satisfy the stronger assumptions of strict convexity and strict monotonicity.

The example we develop is as follows:

$$U(x_1, x_2, x_3) = \begin{cases} (x_1 - 10)^{0.1} (x_2 - 10)^{0.1} (x_3)^{0.8} & \text{if } x_1 \geq 10 \text{ and } x_2 \geq 10 \\ \min\{x_1 - 10, x_2 - 10\} & \text{otherwise} \end{cases}$$

$$M = 2000,\ R = 408,\ p_1 = p_2 = 0,\ p_3 = 1,\ t_1, t_2 \geq 0,\ t_3 = 0.$$

For expediency, we are (first) making an extreme assumption that goods 1 and 2 are publicly provided and there is no private price other than the tax: $p_1 = p_2 = 0$. The entire price for good 1 is the tax t_1 and the entire price of good 2 is the tax t_2. Government Revenue is a constant amount for simplification purposes, in reality the Government would have some kind of profit function to cover the upkeep costs for anything provided. However in this case, we assume that the costs are fixed, this allows us to relax our assumptions later.

For example, good 1 is rounds of golf at a public golf course and good 2 is rides on the subway; i.e., goods 1 and 2 are publicly provided private goods. Later, we relax this extreme assumption and show that our results are qualitatively the same. Currently, however, this assumption of publicly provided private goods allows an enormous simplification of the tax revenue constraint, as we will soon show.

Disposable income (after the consumer buys 10 units each of goods 1 and 2) is

$$2000 - 10t_1 - 10t_2.$$

It is well known (and straightforward to derive) that the demand functions that result from Stone-Geary preferences have a simple form, whenever the consumer has enough money to afford the necessary quantities $\bar{x}_i$'s. For good i, the consumer buys the necessary quantity $\bar{x}_i$ and then spends her disposable income similar to a consumer with Cobb-Douglas preferences. Here, we get:

$$x_1^* = 10 + \frac{0.1(2000 - 10t_1 - 10t_2)}{t_1}$$
$$x_2^* = 10 + \frac{0.1(2000 - 10t_1 - 10t_2)}{t_2}$$
$$x_3^* = 0.8(2000 - 10t_1 - 10t_2)$$

Thus, the government revenue requirement is

$$t_1x_1^* + t_2x_2^* = 408$$
$$\Updownarrow$$
$$10t_1 + 200 - t_1 - t_2 \; +$$
$$10t_2 + 200 - t_1 - t_2 = 408$$
$$\Updownarrow$$
$$t_1 + t_2 = 1.$$

Using the demand functions given above, the table below shows several examples of feasible taxes and resulting consumption and utility for the consumer.

The examples in the table clearly illustrate the lopsided taxes allow the consumer to achieve higher utility. Indeed, because of the simple geometry shown in Figure 2 above, the even taxes are the worst taxes.

	t_1	t_2	x_1^*	x_2^*	x_3^*	U
Scenario A	\$0.50	\$0.50	408	408	1592	1212.51
Scenario B	\$0.55	\$0.45	371.818181	452.222222	1592	1213.73
Scenario C	\$0.45	\$0.55	452.222222	371.818181	1592	1213.73
Scenario D	\$0.90	\$0.10	231.1111	2000	1592	1336.22
Scenario E	\$0.10	\$0.90	2000	231.1111	1592	1336.22

Table 2: Extreme Tax Vectors Preferred to Averages

As an aside, this problem is a bit misbehaved at the boundaries, say if we set $t_1 = \$1$ and $t_2 = \$0$ since we have some issues with division by zero. Also, with one of the goods priced at zero the consumer can consume an infinite amount of that good and achieve infinite utility.

In the following section, we show formally that the Ramsey technique does indeed pick out the even taxes of \$0.50 for each good.

6.2 Ramsey Taxes Are Equal Taxes in Our Example

We next construct the Ramsey tax optimization problem for the Stone-Geary Utility Function given in the prior subsection. First, we express (and simplify) the indirect utility function, using the solutions to the consumer choice problem.

$$V(p_1 + t_1, p_2 + t_2, p_3, m) = 0.1^{0.2} 0.8^{0.8} \left[\frac{m - 10(p_1 + t_1) - 10(p_2 + t_2)}{(p_1 + t_1)^{0.1} (p_2 + t_2)^{0.1} p_3^{0.8}} \right] \tag{3}$$

Note that the numerator in the term in brackets is the disposable income that the consumer has left after buying the necessary amounts (10 each) of goods 1 and 2. In the prior subsection, we have already derived the tax constraint simplifies to: $t_1 + t_2 = 1$.
Thus, the social planner's (Ramsey) tax problem is:

$$\max_{t_1, t_2} \quad V(t_1, t_2, 408) \text{ st } t_1 + t_2 = 1$$

The corresponding Lagrangian is:

$$\mathcal{L} = V(t_1, t_2, 408) + \mu(t_1 + t_2 - 1)$$

Thus, we can see quite clearly that the Kuhn-Tucker conditions expressed in the Ramsey tax problem become

$$
\begin{aligned}
-\lambda x_1^* + \mu &= 0 \\
-\lambda x_2^* + \mu &= 0 \\
t_x + t_y &= 1.
\end{aligned}
$$

where λ is the multiplier from the consumer choice problem (or marginal utility of income) and where we have used Roy's Identity (an envelope theorem) as is standard. The first two of these imply $x_1^* = x_2^*$ or

$$10 + 0.1\left[\frac{M - 10t_1 - 10t_2}{t_1}\right] = 10 + 0.1\left[\frac{M - 10t_1 - 10t_2}{t_2}\right]$$

or simplifying $t_1 = t_2$ and thus along with the tax constraint the unique solution to the Kuhn-Tucker conditions involves $t_1 = t_2 = \$0.50$.

The reason that this example is so nice is because the geometry of the maximization is so familiar to the standard consumer choice problem, with one crucial exception. In particular, the revenue constraint is linear in choice variables, just like the consumer's budget constraint. The difference is that under usual assumption of convex preferences the utility function is quasiconcave, but the indirect utility function is quasiconvex in prices (see Lemma 2): the wrong curvature for the maximization problem.

6.3 Relaxing the Zero Price Assumption

In this section, we consider what happens when the prices of goods 1 and 2 are no longer both zero. To keep things simple, we still suppose that $p_1 = p_2$ and call the price p and we will maintain that p is still close to zero. For the remainder of the example, we will maintain income $m = \$2,000$ and revenue requirement $R = \$408$ and $p_3 = \$1$.

In this case, using (3) the indirect utility function is:

$$V(t_1, t_2, p) = 0.1^{0.2}0.8^{0.8}\left[\frac{\overline{m}(p)}{(p+t_1)^{0.1}(p+t_2)^{0.1}}\right],$$

where $\overline{m}(t_1, t_2, p) = 2000 - 10(t_1 + t_2) - 20p$ is the disposable income.

Logging the indirect utility function (monotonic transformation and no effect on indifference curves) and using the implicit function theorem gives the slope of indifference curves as:

$$-\left[\frac{10}{\overline{m}(t_1,t_2,p)}+\frac{0.1}{p+t_1}\right]\Big/\left[\frac{10}{\overline{m}(t_1,t_2,p)}+\frac{0.1}{p+t_2}\right]$$

and so the slope of any indifference curve is -1 when $t_2 = t_1$.

The tax revenue function is:

$$\begin{aligned} R(t_1,t_2,p) &= t_1\left[10+0.1\frac{\overline{m}(t_1,t_2,p)}{p+t_1}\right]+t_2\left[10+0.1\frac{m}{p+t_2}\right] \\ &= \underbrace{10(t_1+t_2)}_{I}+\underbrace{0.1\overline{m}(t_1,t_2,p)}_{II}\underbrace{\left[\frac{t_1}{p+t_1}+\frac{t_2}{p+t_2}\right]}_{III}. \end{aligned}$$

We can find the slope of any iso-tax-revenue curve using the implicit function theorem as:

$$-\left[10-III+II\frac{p}{(p+t_1)^2}\right]\Big/\left[10-III+II\frac{p}{(p+t_2)^2}\right]$$

which also has a slope of -1 when $t_1 = t_2$.

We next sketch an argument to show that the revenue function is quasiconcave in (t_1, t_2) for fixed price p for goods 1 and 2 for taxes not too extreme such that II is positive and $10 - III$ is positive, so that slope of iso-tax-revenue curves are negative in this region.

- Fact 1: For p small, tax revenue is increasing along 45-degree line ($t_2 = t_1$).
- Fact 2: I and II only depend on the sum of t_1 and t_2.
- Fact 3: For any small enough positive taxes s, t and small fixed price p such that $\overline{m}(s, t, p)$ is positive, we have $R(s, t, p) = R(t, s, p)$.
- Fact 4: For any $p > 0$, term III is strictly concave in (t_1, t_2) (which follows immediately from negative definite Hessian with zero off-diagonals).
- Fact 5: If we take any weighted average (using weight $\lambda > 0$ of tax vectors (s, t) and (t, s), we have sum of the weighted average tax vector is $s + t$.
- Fact 6: If we take any weighted average of tax vectors (s, t) and (t, s) using weight

$\lambda \in (0,1)$,we have:

$$\lambda R(s,t,p) + [1-\lambda]R(t,s,p) < R(\lambda s + [1-\lambda]t, \lambda t + [1-\lambda]s, p).$$

- Fact 7: Iso-tax revenue function is quasiconcave in taxes in this region. The idea is that if we are considering some utility level α then any tax vector in the better than region can be gotten as a convex combination of some pair of tax vectors (s,t) and (t,s) each yielding utility α.

Below we give the figure of when $p_1 = p_2 = \$0.01$. We solved the revenue constraint numerically with equal taxes to get $t_1 = t_2 = 0.80627$ so that the total price for each of the two taxed goods including the penny price is 0.81627 and this yields maximized utility of 1090.37. Alternatively, for $s = 0.50$ and $t = 1.1964331$, but for ease of calculation let us suppose $s = 0.50$ and $t = 1.20$. This will give disposable income of \$1982.80 which gives the (near) optimal quantities $x_1 = 398.78$ and $x_2 = 173.86$ and $x_3 = 1586.21$. These quantities cost the consumer \$1999.96 and raise tax revenue of \$ we get revenue of 408 and indirect utility of: 408.02, just overshooting the revenue requirement, but still yielding utility of 1098.26 doing better for the consumer (and if the consumer spent her last few pennies, slightly more revenue would be raised and slightly more utility would be obtained). In any case, the numbers are really meant to show, once again, that the equal taxes found by the tangency condition (which would result from solving the Kuhn-Tucker conditions from the optimal tax problem) do not guarantee a solution to the optimal tax problem.

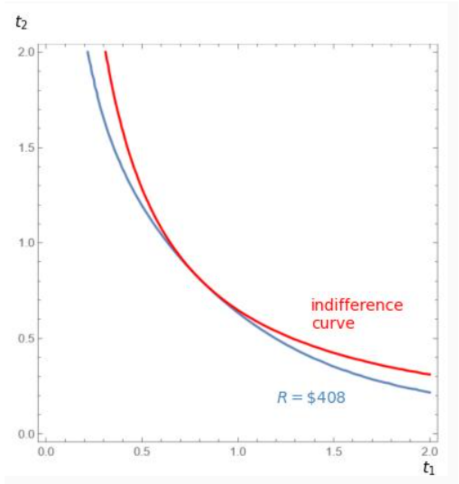

Figure 3: Suboptimal Taxes with Penny Price

In Figure 3, you will see that the tangency occurs at $t_1 = t_2 \approx \$0.82$, with the indifference curve being inside of the $R = \$408$ iso-tax-revenue curve. Any nearby point on the iso-tax-revenue curve will be payoff improving, lying below the shown indifference curve, recalling indirect utility is decreasing in each tax. Now we cannot generally rule out that additional tangencies of some indifference curve and the iso-tax-revenue curve may occur and that perhaps one of those (which in this example will involve lopsided taxes) may indeed be the optimum. However, our example show that a solution to the Kuhn-Tucker conditions need not be a solution to the optimal tax problem; i.e., because of the curvature issues in the objective and constraint functions, Kuhn-Tucker conditions are not sufficient for an optimum. The graph indicates that no tangencies exist other than along the 45-degree line, but for taxes too lop-sided we are not sure of the properties of the iso-tax-revenue curve.

7 Discussion and Conclusion

Although both of our examples with goods provided by the government or goods provided privately but with very low prices were rather contrived, we believe they point the way to better understanding the issues. Even though astronomers knew that other planets outside our solar system must exist, it took a very long time before they found the first such planet. Once the first planet was located, it did not take long before hundreds of other planets were located. Our metaphor is that we have isolated an issue that we always believed invalidated these techniques and even the usual results about spreading taxes across many goods; that issue being the quasiconvexity of indirect utility in prices. It is our hope that now the issue has been identified with this rather contrived pair of examples, more insight can be developed in future research.

Our problem is linked to some other ideas in economics. The first of which being price discrimination. With a lopsided tax, price conscious consumers are able to get around the tax while the richer, less price sensitive absorb the higher taxes, ultimately raising the revenue needed. Our examples, although contrived, may also pose as a potential proxy for lopsided taxes effect on consumer's utility. If used on a large scale, our argument could pose a problem for the widely accepted point of view that any excise tax or carbon tax would create too many distortions in the market, when rather a lopsided tax on polluting goods may be an optimum scenario. Not only optimizing the individual's utility, but also discouraging environmentally damaging practices. Taxing goods with negative externalities and creating a sort of double dividend, by raising government revenue and allowing lower taxes on other goods while discouraging consumption of dirty goods that negatively impact health and the environment. Although some of the literature on the double dividend of pollution taxes has warned that there additional effects which stunt the utility increases obtainable by such pollution taxes; for example, see Bovenberg and de Mooij (1994). Our argument augments the case for taxing dirty goods.

In addition, it is important to address the "fairness" of a lopsided tax. Various industries and corporations are bound to put up a fight when facing large lopsided taxes. However, in favor of "fairness" these taxes can be rotated in some way, while the consumer retains buying power of some products. Again, however, a lopsided tax can be used in conjunction with taxing dirty or other sinful goods. For example, take the excise tax on cigarettes: the social planner maximizes individual utility and raises government revenue by taxing something that is scientifically known to cause serious illness. Although incredibly simplified, the cigarette tax a good example of a lopsided tax option used to discourage smoking while also incentivizing better health choices and raising required revenue.

Ultimately, tax research can only take us so far. This paper has allowed us to dig into a niche portion of tax literature, and has demonstrated to us that we know much less about taxes than we would like. Maybe the only sensible thing we can say about taxes is that they should be small (but perhaps not uniformly small!), that only the most important things should be bought with the government revenue so that not much revenue is required.

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