THE FIRST ANNUAL (2006) KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. A shopper buys 3 apples and 2 oranges and pays \$1.78. Changing his mind, he exchanges an orange for another apple and has to pay an additional 16¢. What is the price of a single apple?

   (A) 26¢ (B) 38¢ (C) 42¢ (D) 48¢ (E) 54¢

2. Which of the following numbers is largest?

   (A) $2+\sqrt{3}+\sqrt{5}$ (B) $2+\sqrt{8}$ (C) $3+\sqrt{7}$ (D) $4+\sqrt{5}$ (E) $\sqrt{10}$

3. Let $w = \frac{x_1}{|x_1|} + \frac{x_2}{|x_2|} + \frac{x_3}{|x_3|} + \ldots + \frac{x_{10}}{|x_{10}|}$. If $x_1, x_2, x_3, \ldots, x_{10}$ are all non-zero real numbers, how many distinct values can w have?

   (A) 10 (B) 11 (C) 20 (D) 21 (E) 22

4. The sum of the measures of the first three interior angles of a pentagon is 345. The measure of the fourth angle is the average of the measures of the first three. Compute the number of degrees in the measure of the fifth angle.

   (A) 72 (B) 80 (C) 108 (D) 109 (E) 120

5. Angie rolls one ordinary six-sided die repeatedly, keeping track of each number she rolls, and stopping as soon as any number is rolled for the third time. If Angie stops after her twelfth roll, and the sum of these rolls is 47, which number has occurred three times?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Continued on back

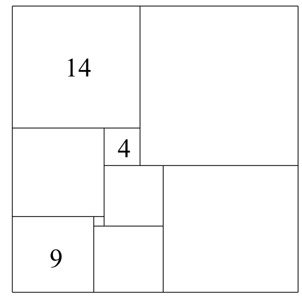
6. When Mom came home and found the cookie jar broken, she gathered up her four children for an explanation and the following discussion took place:

Ann: "I didn't do it."
Bob: "I didn't do it and Ann didn't do it."
Cal: "I didn't do it and Bob didn't do it."
Deb: "Ann didn't do it and Bob didn't do it."

Mom later found out that exactly one of the above four statements was false, the rest being true. Which one of the children broke the cookie jar?

(A) Ann (B) Bob (C) Cal (D) Deb (E) Cannot be determined

7. A *squared rectangle* is a rectangle whose interior can be divided into two or more squares. An example of a squared rectangle is shown. The number written inside a square is the length of a side of that square. Compute the area of the squared rectangle shown.



(A) 1024 (B) 1056 (C) 1089 (D) 1120 (E) 1122

8. A *prime-prime* is a prime number that yields a prime when its units digit is omitted. (For example, 317 is a three-digit prime-prime because 317 is prime and 31 is prime). How many two-digit prime-primes are there? (Recall that 1 is not a prime number.)

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

9. The function f has the property $f(x) = 1 - f(x - 1)$. If $f(2) = 12$, compute $f(2006)$.

(A) 0 (B) 12 (C) 2006 (D) 2018 (E) None of these

10. One root of the equation $2x^3 - 5x^2 - 8x + d = 0$ is the negative of another. Compute d.

(A) -10 (B) -4 (C) 8 (D) 20 (E) 24

11. The lengths of three consecutive sides of a quadrilateral are equal. If the angles included between these sides have measures of 60 degrees and 70 degrees, what is the measure of the largest angle of the quadrilateral?

(A) 145° (B) 150° (C) 155° (D) 160° (E) 165°

12. To number the pages of a mathematics textbook (beginning with page 1), the printer used a total of 2541 digits. How many pages did the book contain?

(A) 880 (B) 881 (C) 882 (D) 883 (E) 884

Continued on next page

13. If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, compute the probability that this number will be divisible by 11.

(A) $\frac{1}{5}$ (B) $\frac{1}{7}$ (C) $\frac{1}{11}$ (D) $\frac{1}{14}$ (E) $\frac{1}{15}$

14. A square is sketched on the coordinate plane so that its sides have slopes of $\frac{1}{4}, -4, \frac{1}{4}$, and -4, respectively. One of the diagonals has a positive slope. Compute this slope.

(A) $\frac{5}{3}$ (B) $\frac{3}{5}$ (C) $\frac{9}{7}$ (D) $\frac{9}{13}$ (E) $\frac{13}{9}$

15. A group of twenty-five coins, whose total value is \$2.75, is composed of nickels, dimes, and quarters. If the nickels were dimes, the dimes were quarters, and the quarters were nickels, the total would be \$3.75. How many quarters are there in the collection?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

16. If R is the remainder when each of the numbers 1059, 1417, and 2312 is divided by D, where D is an integer greater than 1, compute the value of D – R.

(A) 11 (B) 15 (C) 19 (D) 23 (E) 27

17. One term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{14}$ is linear (i.e. degree = 1). Compute the numerical coefficient of this term.

(A) 3432 (B) 3003 (C) 2002 (D) 1001 (E) 364

18. Let A, B, and C represent distinct digits. A four-digit positive integer of the form ABCA has the property that the two-digit integers AB, BC, and CA are all prime. Compute the number of all such four-digit integers ABCA.

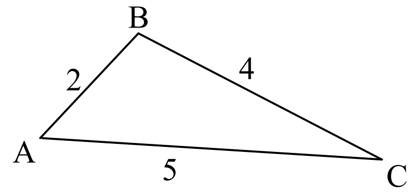
(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

19. Andy was at a party and knew that 3 of the guests were born on the same day of the week <u>and</u> in the same month of the year. He also knew that if <u>any one</u> guest left the party (including himself), this would no longer be true. How many people (including Andy) were at the party?

(A) 120 (B) 144 (C) 169 (D) 184 (E) 196

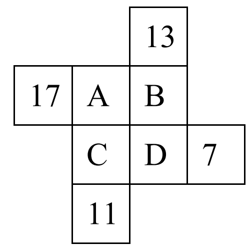
Continued on back

20. In triangle ABC, AB = 2, BC = 4, and AC = 5. Compute the value of $\frac{\sin A - \sin B}{\sin C}$.



(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $\frac{2}{5}$ (D) $-\frac{2}{5}$ (E) 1

21. In the pattern shown, A, B, C, and D are prime numbers and all eight numbers are different. The sum along any 3-number row or column is the same number S. Compute the smallest possible value of S.



(A) 67 (B) 73 (C) 77 (D) 81 (E) 83

22. A three-digit number has the following interesting property. If the middle digit is deleted, the remaining two-digit number is the square of the deleted digit. Find the sum of all such three-digit numbers? (Note: Numbers like 007 and 039 are not considered three-digit numbers.)

(A) 2137 (B) 2342 (C) 2566 (D) 2675 (E) 2821

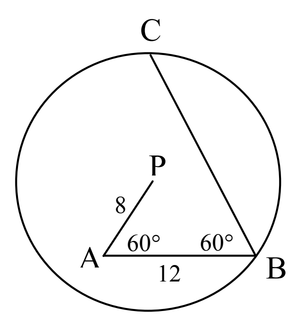
23. Compute the sum of all integral values of x, $0° < x < 90°$, for which $\sin(x) = \sin(x^2)$.

(A) 80 (B) 82 (C) 117 (D) 126 (E) 161

24. The twentieth term of an arithmetic sequence is log(20) and the thirty-second term is log(32). Exactly one term of the arithmetic sequence is a rational number. What is that rational number? (Logarithms are to base 10.)

(A) $\frac{5}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{8}{5}$ (E) $\frac{7}{4}$

25. In circle P, PA = 8 inches and AB = 12 inches. The measures of angles A and B are 60° each. Compute the number of inches in the length of chord BC.



(A) $8\sqrt{3}$ (B) 12 (C) $12\sqrt{3}$ (D) 16 (E) 20

END OF CONTEST

SOLUTIONS – KSU MATHEMATICS COMPETITION – PART I – 2006

1. C We are given $3A + 2O = 1.78$ and $4A + O = 1.94$. Solving these two equations together for A, we obtain A = \$0.42 or **42¢**.

2. D $2+\sqrt{8} = 2+2\sqrt{2} = 2+\sqrt{2}+\sqrt{2}$, therefore, choice A is larger than choice B. Since $2+\sqrt{3}+\sqrt{5} < 2+\sqrt{4}+\sqrt{5} = 4+\sqrt{5}$, choice D is larger than choice A. Choice D is also larger than choice E since $\sqrt{10} < 4$. Since $\sqrt{5} > 2$, choice D is larger than $4 + 2 = 6$. Since $\sqrt{7} < 3$, choice C is less than $3 + 3 = 6$. Therefore, choice D is larger than choice C. Hence, choice **D** is the largest.

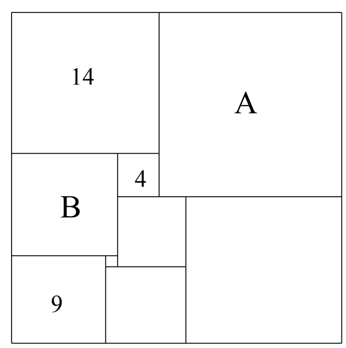
3. B If all x_i are positive then $w = 10$. If all x_i are negative then $w = –10$. If only one of the x_i is positive then $w = –9 + 1 = –8$. Similarly, if only one of the x_i is negative then $w = 8$. Following this pattern, the only possible values for w are –10, –8, –6, –4, –2, 0, 2, 4, 6, 8, 10, for a total of **11** possible values of w.

4. B The measure of the fourth angle is $\frac{345}{3} = 115$. Therefore, the measure of the fifth angle is $540 – (345 + 115) =$ **80**.

5. E After the eleventh roll, no number had yet appeared three times. This means that five of the numbers appeared twice, and the other number once. Call this number A. If A was rolled on the twelfth roll, the total would be $2(1+2+3+4+5+6) = 42$. Therefore, the total after 11 rolls is $42 – A$. If B is the number rolled for the third time, then $42 – A + B = 47$ and so $B – A = 5$. The only possible value for B is **6**.

6. C If Ann "did it", then both Ann and Bob made false statements. Therefore, Ann didn't do it. Similarly, If Bob did it, then both Cal and Bob made false statements, so Bob didn't do it. Thus, both Ann and Bob made true statements, which means Deb's statement is also true. Hence, Cal made the false statement and since we already know Bob didn't do it, **Cal** did.

7. B Clearly, $A = 14 + 4 = 18$, making the length of the top of the squared rectangle $14 + 18 = 32$. Similarly, $B = 14 – 4 = 10$, making the length of the left side of the squared rectangle $14 + 10 + 9 = 33$. Therefore, the area of the squared rectangle is $(33)(32) =$ **1056.**



8. C For a two-digit number to be a prime-prime, its ten digit must be a prime. Thus only numbers in the 20s, 30s, 50s, and 70s are eligible. The primes in each of these groups are: 23, 29, 31, 37, 53, 59, 71, 73, 79, for a total of **9**.

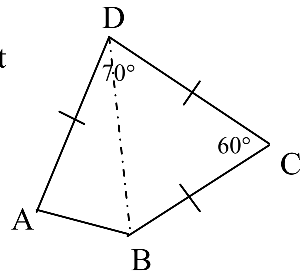
9. B Since $f(x) = 1 – f(x – 1)$, $f(x + 1) = 1 – f(x)$. Therefore, $f(x + 1) = 1 – [1 – f(x – 1)] = f(x – 1)$. Thus $f(2) = f(4) = f(6) = \ldots = f(2006) =$ **12**.

10. D Let r and -r be the two roots. Substituting both we obtain

$$2r^3 - 5r^2 - 8r + d = 0 \text{ and } -2r^3 - 5r^2 + 8r + d = 0$$

Adding the two equations gives $-10r^2 + 2d = 0 \Rightarrow d = 5r^2$
Subtracting the two equations gives $4r^3 - 16r = 0 \Rightarrow 4r(r^2 - 4) = 0 \Rightarrow r = \pm 2$
Therefore, $d = 5(2^2) = \mathbf{20}$.

11. A Draw the segment DB. Note that triangle BDC is equilateral, so that angles BDC and CBD measure 60°, angle ADB measures 10° and BD = AD, making ΔABD isosceles. Therefore, the measures of angle DAB and DBA are 85°, and measure of angle CBA is 60 + 85 = 145°. Hence, the measure of the largest angle is **145°**.



12. D From pages 1-9 9 digits used
From pages 10-99 $90 \cdot 2 = 180$ digits used

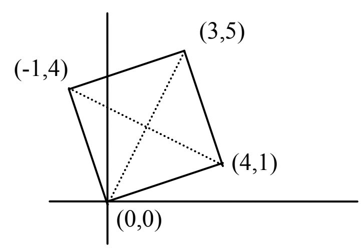
189 total digits used for pages 1-99.

2541-189 = 2352 digits available for use as three-digit page numbers.

2352÷3 = 784 pages. Total 784 + 99 = **883** pages.

13. A Since the largest possible digit in base 10 is 9, the sum of the five digits can be at most 45. The given sum, 43, can come about in the following ways.
(i) One of the digits is 7, all others are 9. There are five such possibilities: 79999, 97999, 99799, 99979, 99997.
(ii) Two of the digits are 8, the other three are 9. This can happen in ${}_5C_2 = 10$ ways. 88999, 89899, 89989, 89998, 98899, 98989, 98998, 99889, 99898, and 99988.
Now, a number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. For example, the five-digit number ABCDE is divisible by 11 if A – B + C – D + E is divisible by 11. Thus, only three of the 15 possibilities, namely 97999, 99979, and 98989 are divisible by 11. Therefore, the required probability is $\frac{3}{15} = \mathbf{\frac{1}{5}}$.

14. A The quickest method here is to create a graphic model which matches the description. One candidate is shown at the right. The slopes of the diagonals are $\frac{4-1}{-1-4}$ and $\frac{5-0}{3-0}$ or $-\frac{3}{5}$ and $\frac{5}{3}$.



15. C Before the coins are switched, we have 5N + 10D + 25Q = 275. After the switch, we have 10N + 25D + 5Q = 375. Doubling the first equation and subtracting the second equation gives 9Q – D = 35 or D = 9Q – 35. Looking for pairs of values (Q,D) that satisfy this equation, and keeping in mind the original amount of money given ($2.75), we have only two possible pairs: Q =4, D = 1 and Q = 5, D = 10. Using the second of our two original equations, the first of these pairs gives N = 33, which is more money than allowed. Therefore, Q = **5**.

16. B Since each of the given numbers 1059, 1417, and 2312, when divided by D, has the same remainder, D divides the differences between the numbers. Factoring the differences,

$$2312 - 1417 = 895 = 5 \cdot 179$$
$$1417 - 1059 = 358 = 2 \cdot 179$$

Since 179 is prime, D = 179. Now $1059 = 5 \cdot 179 + 164$, thus, R = 164. Therefore, D – R = 179 – 164 = **15**.

17. C Using the Binomial Theorem, let $\binom{14}{n}\left(x^2\right)^m\left(\frac{1}{x}\right)^n = \binom{14}{n}\left(x^{2m}\right)\left(\frac{1}{x^n}\right) = \binom{14}{n}x^{2m-n}$ be the n^{th} term of the expansion., where m + n = 14. Since the linear term has degree 1, 2m - n = 1. Adding these two equations and solving yields m = 5 and n = 9. Therefore the desired coefficient is

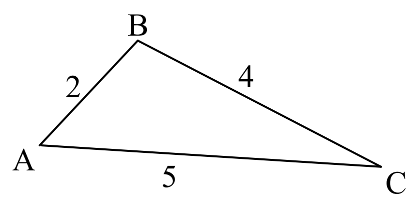
$$\binom{14}{9} = \frac{14!}{(5!)(9!)} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 13 \cdot 11 \cdot 2 = \mathbf{2002}$$

18. D Since AB, BC, and CA are all primes, then A, B, and C must be drawn from {1, 3, 5, 7, 9}. If A = 1, then B could be 3, 7, or 9. If B = 3, then C = 7 but not 9 (since 39 is not prime), giving 1371. If B = 7, then C = 3 since if C = 9, then the number ends with 91 which is not prime. Thus, 1731 works. Continuing in this manner we obtain the following solutions: 1371, 1731, 1971, 3173, 3713, 7137, 7197, 7317, 9719 for a total of **9** solutions.

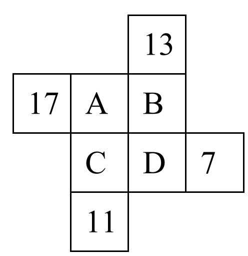
19. C The greatest number of people a group can have with <u>no two</u> born in the same month and on the same day of the week is (12)(7) = 84. The greatest number of people a group can have with <u>no three</u> born in the same month and on the same day of the week is 2(84) = 168. Add one more person, and there must be at least 3 born in the same month and on the same day. Therefore, there must have been **169** guests at the party.

20. B $\frac{\sin A - \sin B}{\sin C} = \frac{\sin A}{\sin C} - \frac{\sin B}{\sin C}$. Using the Law of Sines,

$\frac{\sin A}{\sin C} = \frac{4}{2}$ *and* $\frac{\sin B}{\sin C} = \frac{5}{2}$, so that $\frac{\sin A - \sin B}{\sin C} = -\mathbf{\frac{1}{2}}$.



21. A Since the first row has A + B + 17, all other rows and columns must also have that sum. Since A + B + 17 = B + D + 13, then D = A + 4. Since A + B + 17 = A + C +11, then C = B + 6. The smallest prime for A that will make D prime is A = 19. When A = 19, D = 23, for which B = 31, C = 37. For these values, the required sum S is 67. The smallest choices for B and C are B = 23 and C = 29. However, these require A = 37 and D = 41, which yields a higher sum, S = 77. Therefore, the smallest value for the sum, S, is **67**.



22. E The only possibilities are three digit numbers for which the middle digit has a two digit square: 4, 5, 6, 7, 8, and 9. The three digit numbers are 146, 255, 366, 479, 684, and 891. Their sum is **2821**.

23. E Any value x, $0° < x < 90°$, which satisfies $\sin(x) = \sin(x^2)$ must satisfy one of the following three equations:

(i) $x^2 = x$

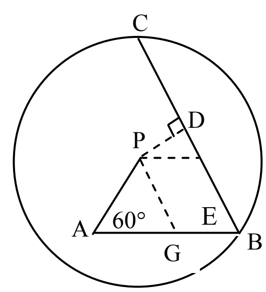
(ii) $x^2 - x = 360m$, where m is a positive integer

(iii) $x^2 + x = (2n + 1)180$, where n is a positive integer.

For equation (i), only x = 1 works.
For equation (ii), $x^2 - x = x(x-1) = 360m = 2^3 \cdot 3^2 \cdot 5m$. Therefore, we need two consecutive integers whose product contains $2^3 \cdot 3^2 \cdot 5$. The only such integers are $x - 1 = 80$ ($2^5 \cdot 5$) and $x = 81$ (3^4).
For equation (iii), $x^2 + x = x(x+1) = (2n+1)180 = (2n+1)(2^2 \cdot 3^2 \cdot 5)$. Therefore, we need two consecutive integers whose product is an odd multiple of $2^2 \cdot 3^2 \cdot 5$.
There are two such pairs: $x = 35$ ($5 \cdot 7$) and $x + 1 = 36$ ($2^2 \cdot 3^2$), and $x = 44$ ($11 \cdot 2^2$) and $x + 1 = 45$ ($3^2 \cdot 5$).
Thus, the only values of x ($0° < x < 90°$) which satisfy the equation are 1, 35, 44, and 81 and there sum is **161**.

24. A $\text{Log}(20) = \log(10 \cdot 2) = 1 + \log 2$ and $\log(32) = \log(2^5) = 5\log(2)$. Therefore, the common difference for this arithmetic progression is $\dfrac{5\log(2) - [1 + \log(2)]}{12} = \dfrac{4\log(2) - 1}{12} = \dfrac{1}{3}\log(2) - \dfrac{1}{12}$.
Therefore, the 17[th] term of the progression is the one that is rational.
$[1 + \log(2) - 3[\frac{1}{3}\log(2) - \frac{1}{12}] = 1 + \log(2) - \log(2) + \frac{3}{12} = \mathbf{\frac{5}{4}}$.

25. E (*Method 1*) Construct PG // CB (G on AB), and PE//AB (E on BC). Then ΔAPG is equilateral with AG = PG = 8 and GB = 4. PEBG is a parallelogram, so that PE = 4 and BE = 8. Construct PD $\perp$ BC (D on BC). Noting that ΔPDE is a 30-60-90 triangle, ED = $\frac{1}{2}$(PE) = 2, and BD = 8 + 2 =10. Since PD bisects BC, BC = **20**.



(*Method 2*) Extend AP through P to E on BC. ΔABE is equilateral with AB = BE = AE = 12, and PE = 4. Construct PD $\perp$ BC (D on BC). Since ΔPDE is a 30-60-90 triangle, ED = $\frac{1}{2}$(PE) = 2, and BD = 12-2 = 10. Since PD bisects BC, BC = **20**.