THE 2007–2008 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. How many three digit positive integers are there such that the sum of the digits is a multiple of 7, the first two digits add to 12, and the number contains a repeated digit?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

2. Which of these numbers is the average (mean) of the other four?

   (A) 27 (B) 36 (C) 25 (D) 29 (E) 28

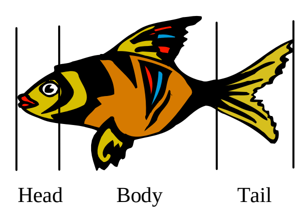
3. If $\dfrac{a+2b}{a-2b}=3$, what is the value of $\dfrac{a+3b}{a-3b}$?

   (A) 7 (B) 6 (C) 4 (D) 2 (E) None of these

4. The sum of the lengths of three of the four sides of a rectangle is 2007. The sum of the length of the fourth side and the length of a diagonal of the rectangle is also 2007. What is the ratio of the length of the longer side to the length of the shorter side of this rectangle.

   (A) $\sqrt{2}:1$ (B) $\sqrt{3}:1$ (C) 2:1 (D) 3:1 (E) 4:1

5. A fish had a tail as long as its head plus a quarter the length of its body. Its body was three-fourths of its total length. If its head was 4 centimeters long, what was the entire length of the fish?



   (A) 100 cm (B) 120 cm (C) 128 cm (D) 132 cm (E) 136 cm

Continued on back

6. What is the value of $\log\frac{1}{2}+\log\frac{2}{3}+\log\frac{3}{4}+\log\frac{4}{5}+...+\log\frac{99}{100}$?

(A) 0 (B) 1 (C) 2 (D) -1 (E) -2

7. Two people take turns rolling a die. What is the probability that the second person will roll a 1 before the first person rolls a 6?

(A) $\frac{1}{2}$ (B) $\frac{5}{11}$ (C) $\frac{7}{12}$ (D) $\frac{13}{36}$ (E) $\frac{6}{11}$

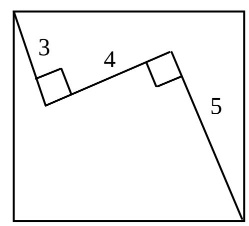
8. There are 35 sets of twins at the "Twins-R-Us" Day Care center. Among these children, there is a total of 38 boys and there are four more sets of girl-girl twins than girl-boy sets of twins. How many boy-boy sets of twins are there?

(A) 8 (B) 10 (C) 12 (D) 15 (E) 19

9. A, B, and C are three sets. $A\cup C = \{1,2,3,4,5,6\}$, $B\cup C = \{1,2,3,4\}$ $A\cap C = \phi$, $A\cap B = \{3\}$, and $B\cap C = \{1,2\}$. Find B.

(A) {1,2} (B) {1,3} (C) {1,2,3} (D) {1,2,4} (E) {1,2,3,4}

10. To the nearest tenth, what is the area of the square shown?



(A) 40.0 (B) 42.5 (C) 45.0 (D) 47.5 (E) 49.0

11. Suppose $a_1, a_2, a_3, ..., a_k$ form an arithmetic sequence. If $a_5 + a_8 + a_{11} = 10$, and $a_7 + a_{10} + a_{13} = 12$, and $a_k = 11$, compute k.

(A) 17 (B) 19 (C) 23 (D) 29 (E) 31

12. Consider the set of integers {1000, 1001, 1002, …1998, 1999, 2000}. There are times when a pair of consecutive integers in this set can be added without "carrying". For example 1213 + 1214 requires no carrying, while 1217 + 1218 does require carrying. For how many pairs of consecutive integers in the set is <u>no</u> carrying required when the two numbers are added? (Note: 1213 + 1214 and 1214 + 1213 should <u>not</u> be considered different pairs.)

(A) 156 (B) 162 (C) 169 (D) 175 (E) 196

Continued on next page

13. If x and n are positive integers such that $x^2 + 615 = 2^{2n}$, what is the value of x + n?

(A) 61 (B) 63 (C) 65 (D) 67 (E) 69

14. Let d represent the length of the diagonal of a cube. Which of the following represents the surface area of the cube?

(A) $d^2\sqrt{2}$ (B) $d^2\sqrt{3}$ (C) $\frac{3}{2}d^2$ (D) $2d^2$ (E) $3d^2$

15. Find the sum of all values of x which satisfy: $\frac{1}{x^2-38x-29}+\frac{1}{x^2-38x-45}=\frac{2}{x^2-38x-69}$.

(A) 29 (B) 38 (C) 45 (D) 69 (E) None of these

16. The number 2007 has N factors (including itself and 1). Compute the number of two-digit positive integers which have exactly N factors.

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

17. Rectangle ABCD is placed on a coordinate plane so that the coordinates of A, B, and C, respectively, are (1, 5), (7, 9), and (9, 6). A line through the origin divides the rectangle into two regions with equal areas. What is the slope of this line?

(A) $\frac{7}{4}$ (B) $\frac{11}{10}$ (C) $\frac{15}{16}$ (D) 1 (E) 2

18. John travels from point P to point Q in 8 minutes. Mary travels from Q to P along the same route. They start at the same time and each travels at a constant rate. If Mary reaches point P 18 minutes after they meet, how many minutes did the entire trip take Mary?

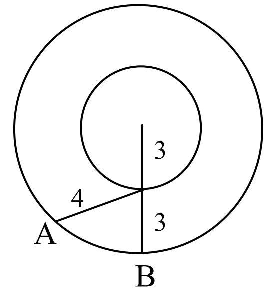
(A) 20 (B) $22\frac{1}{2}$ (C) 24 (D) $25\frac{1}{4}$ (E) 26

19. Compute the number of positive integers *a* for which there exists an integer *b*, $0 \le b \le 2007$, such that both $x^2 + ax + b$ and $x^2 + ax + b + 1$ have integer solutions.

(A) 40 (B) 41 (C) 42 (D) 43 (E) 44

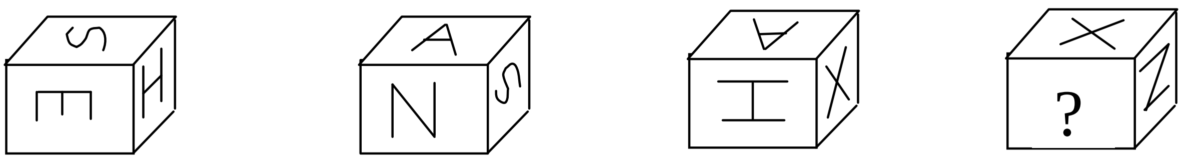
Continued on back

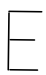
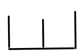
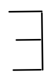
20. Two concentric circles are shown. The radius of the inner circle is 3, and the distance between the circles is 3. A line segment of length 4 has its endpoints on both circles. Compute the distance from point A to point B .



(A) $\sqrt{7}$ (B) $\sqrt{14}$ (C) $\sqrt{15}$ (D) $\sqrt{19}$ (E) 5

21. Below are four different views of the same toy alphabet block. Which of the following should appear on the blank (where the **?** is).



(A)  (B)  (C)  (D)  (E) 

22. Let P(x) = $x^4 + ax^3 + bx^2 + cx + d$. If P(1) = 10, P(2) = 20, and P(3) = 30, compute the value of P(10) + P(-6).

(A) 4896 (B) 5240 (C) 6064 (D) 7816 (E) 8104

23. Of the animals entered in a dog show, the number of poodles is at least one-fifth of the number of beagles and at most one-sixth the number of collies. The number of dogs which are poodles or beagles is at least 23. What is the minimum number of collies entered in this show?

(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

24. It is possible to place positive integers into the twenty-one vacant squares of the 5x5 square shown at the right, so that the numbers in each row and each column form arithmetic sequences. What number must occupy the square marked by the asterisk ($*$).

			*	
	74			
				186
		103		
0				

(A) 118 (B) 126 (C) 134 (D) 142 (E) 150

25. One vertex of an equilateral triangle lies on the point with coordinates (1, 4). The other two vertices lie on the line whose equation is y = 3x – 4, at the points (x_1, y_1) and (x_2, y_2). Compute the sum $y_1 + y_2$.

(A) 7 (B) 7.5 (C) 8 (D) 8.5 (E) None of these

END OF CONTEST

THE 2007–2008 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

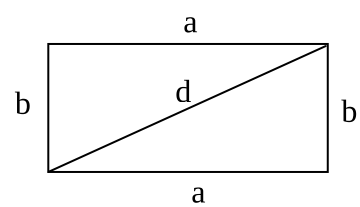
Part I – Solutions:

1. E Let *a*, *b*, and *c* be the digits. The sum of the digits must be 14 or 21. Thus $a + b + c = 14$ or $a + b + c = 21$. Since $a + b = 12$, $c = 2$ or $c = 9$. If $c = 2$, then *a* and *b* must both be 6. If $c = 9$, then *a* and *b* could both be six, or one could be 9 and the other 3. Hence the possibilities are 662, 669, 939, 399, for a total of four.

2. D Although the problem can be done by trial and error using the choices, note that if one of the numbers is the mean of the other four, it is the mean of all 5.
$$\frac{27+36+25+29+28}{5} = 29.$$

3. A $\frac{a+2b}{a-2b} = 3 \;\Rightarrow\; a = 4b$. Substituting, $\frac{a+3b}{a-3b} = \frac{4b+3b}{4b-3b} = \frac{7b}{b} = 7$.

4. B We are given $2b + a = 2007$ and $a + d = 2007$. Subtracting the second equation from the first, $2b = d$. Substituting into $a^2 + b^2 = d^2$, we get $a^2 + b^2 = 4b^2$, from which we obtain $a^2 = 3b^2$ and $\frac{a}{b} = \boldsymbol{\frac{\sqrt{3}}{1}}$



5. C Let T = the length of the tail, and let B = length of the body. Then $T = 4 + \frac{B}{4}$ and $B = \frac{3}{4}[B + 4 + (4 + \frac{B}{4})]$. Solving gives B = 96, T = 28 and the entire length of the fish is 96 + 28 + 4 = **128** centimeters.

6. E Using the properties of logarithms, the given expression becomes
$\log 1 - \log 2 + \log 2 - \log 3 + \log 3 - \log 4 + \log 4 - \ldots + \log 99 - \log 100 = -2$.

7. B Let the probability that the first person rolls a 6 first be *x*. If, on her first role, she gets a 6, the second player can't get a 1 first. If she doesn't get a 6, the second player has a probability of *x* that he'll get a 1 first. Therefore, the second player's probability of getting a 1 first is $\frac{5}{6}x$. Since one or the other of these must happen, $x + \frac{5}{6}x = 1$. Hence $x = \frac{6}{11}$ and the probability the second person will roll a 1 before the first person rolls a 6 is $\frac{5}{6}x = \boldsymbol{\frac{5}{11}}$.

8. D If we have *x* girl-boy sets, *x*+4 girl-girl sets, then we have 35-($x + x$ +4) = 31 – 2*x* boy-boy sets. Therefore, we have a total of $x + 2(31 - 2x) = 38$ boys. Solving, we obtain $x = 8$. Hence, there are 31-16 = **15** boy-boy sets.

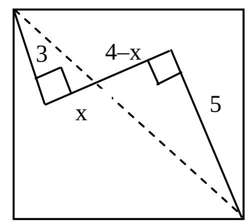
9. C Since $B \cup C = \{1,2,3,4\}$ and $B \cap C = \{1,2\}$, there are only four possible combinations for B and C.

(i) $B = \{1,2,3,4\}$, $C = \{1,2\}$
(ii) $B = \{1,2\}$, $C = \{1,2,3,4\}$
(iii) $B = \{1,2,3\}$, $C = \{1,2,4\}$
(iv) $B = \{1,2,4\}$, $C = \{1,2,3\}$

Since $A \cap B = \{3\}$ and $A \cap C = \phi \Rightarrow$ B contains 3 and C does not. Therefore, we can eliminate (ii) and (iv) as possibilities.
Suppose possibility (i) was correct. Since $A \cup C = \{1,2,3,4,5,6\}$, then $A = \{3,4,5,6\}$ However, this would make $A \cap B = \{3,4\}$, which contradicts the given information.

No such contradictions arise from possibility (iii) above and B = **{1,2,3}**.

10. A Draw the diagonal indicated. The two triangles formed are similar (AA). Therefore, $\frac{3}{x} = \frac{5}{4-x}$ and $x = \frac{3}{2}$. Using the Pythagorean Theorem to find the length of the hypotenuse of each right triangle ($\frac{3\sqrt{5}}{2}$ and $\frac{5\sqrt{5}}{2}$) and adding we find the length of the diagonal of the square is $4\sqrt{5}$. Thus, the area of the square is $\frac{1}{2}d^2 =$ **40**.



11 E $a_5 + a_8 + a_{11} = 3(a_8) = 10$, and $a_8 = \frac{10}{3}$. Similarly, $a_7 + a_{10} + a_{13} = 3(a_{10}) = 12$, and $a_{10} = \frac{12}{3} = 4$. Since a_9 is the average of a_8 and a_{10}, $a_9 = \frac{11}{3}$, so that the common difference $d = \frac{11}{3} - \frac{10}{3} = \frac{1}{3}$. Also, $a_1 = a_9 - 8d = \frac{11}{3} - 8 \cdot \frac{1}{3} = 1$.
Therefore, $a_k = a_1 + (k-1)d \Rightarrow 11 = 1 + (k-1)\frac{1}{3}$ and k = **31**.

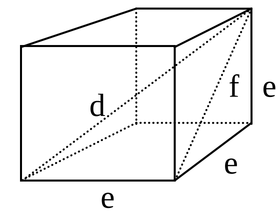
12. A The following list shows sequences of consecutive integers that contain at least one pair of consecutive integers that can be added without carrying and the number of pairs in that sequence that can be added without carrying.

Sequence	Number of pairs	Sequence	Number of pairs
1000-1005	5	1099-1150	31
1009-1015	6	1199-1250	31
1019-1025	6	1299-1350	31
1029-1035	6	1399-1450	31
1039-1045	6	1499-1500	1
1049-1050	1	1999-2000	1
Total	30	Total	126

Grand Total: **156**

13. C We can write 615 = $2^{2n} - x^2 = (2^n)^2 - x^2 = (2^n + x)(2^n - x)$. Thus we have a factorization of 615 into integers. The possible factorizations of 615 are $1 \cdot 615,\ 3 \cdot 205,\ 5 \cdot 123,\ 15 \cdot 41$. But the sum of factors is $2^n + x + 2^n - x = 2^{n+1}$. Only the pair of factors 5, 123 add to a power of 2. Thus, $2^{n+1} = 128 = 2^7$, so that $n = 6$. Then x = 59 and the desired sum is 65.

14. D $d^2 = f^2 + e^2 = e^2 + e^2 + e^2 = 3e^2$. So $e = \dfrac{d}{\sqrt{3}}$.



Since surface area equals $6e^2$, surface area = $6(\dfrac{d}{\sqrt{3}})^2 = 2d^2$.

15. B Let A = $x^2 - 38x$. Then the given equation, $\dfrac{1}{x^2 - 38x - 29} + \dfrac{1}{x^2 - 38x - 45} = \dfrac{2}{x^2 - 38x - 69}$, becomes $\dfrac{1}{A - 29} + \dfrac{1}{A - 45} = \dfrac{2}{A - 69}$. Multiplying both sides by the product of the three denominators gives

$$A^2 - 114A + 3105 + A^2 - 98A + 2001 = 2A^2 - 148A + 2610.$$

Solving, we obtain A = 39. Thus $x^2 - 38x = 39$ or $(x - 39)(x + 1) = 0$ and x = 39, -1. The required sum is **38.**

16. D Since the prime factorization of 2007 = $(3^2)(223^1)$, it has (2+1)(1+1) = 6 factors. Only numbers with prime factorizations of the form $(A)(B^2)$, or A^5 will also have exactly of six factors. For the first form, $(A)(B^2)$, the following chart shows the possible combinations for A and B that will give a two-digit number:

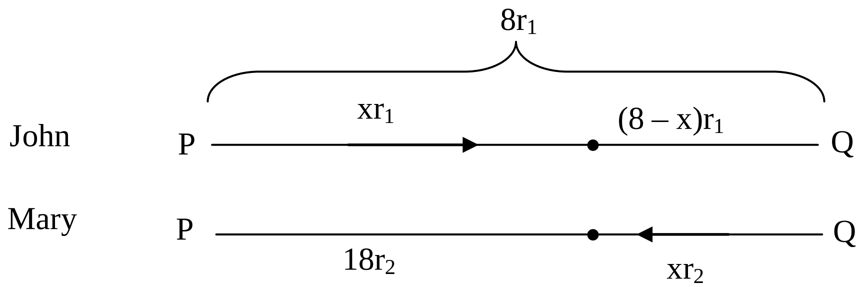
A	B
2	3, 5, 7
3	2, 5
5	2, 3
7	2, 3
11	2, 3
13	2
17	2
19	2
23	2

For the second form, A^5, only A = 2 will give a two-digit number.
Thus, there are a total of **16** possible two-digit numbers with six factors.

17. B Any line that divides a rectangle's area in half passes through the center of the rectangle (i.e the intersection of the diagonals). Since the diagonals of a rectangle bisect each other, this point is the midpoint of diagonal AC. It's coordinates are $\left(\frac{1+9}{2}, \frac{5+6}{2}\right) = \left(5, \frac{11}{2}\right)$.

Thus the slope of the line through this point and (0, 0) is $\frac{11}{10}$.

18. C Let x represent the time it takes John to reach their meeting point, and let r_1 and r_2 represent John's and Mary's rates, respectively. Then the information given, and the distances may be represented as shown in the diagram.



$18r_2 + xr_2 = 8r_1 \Rightarrow r_2 = \frac{8r_1}{18+x}$. Noting that $xr_1 = 18r_2$ we have

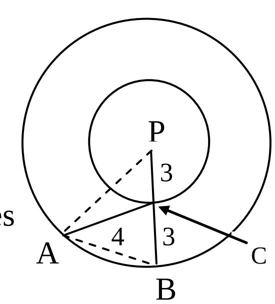
$xr_1 = 18\left(\frac{8r_1}{18+x}\right) = \frac{144r_1}{18+x}$. Dividing this last equation by r_1 we obtain

$x = \frac{144}{18+x} \Rightarrow x^2 + 18x - 144 = 0$.

Solving, we obtain x = -24 (not acceptable), and x = 6. Therefore, Mary travels 6 + 18 = **24 minutes**, to complete the entire trip.

19. E If these polynomials have integer roots then the discriminants $a^2 - 4b$ and $a^2 - 4b - 4$ are perfect squares. The only perfect squares that differ by 4 are 0 and 4. Solving for a, $a = \sqrt{4b+4} = 2\sqrt{b+1}$. Thus b+1 is a perfect square, and since $44^2 = 1936$ and $45^2 = 2025$, the allowed solutions for b are {1-1, 4-1, 9-1, ... 44^2-1}, which means there are **44** solutions.

20. B Draw radius PA (length 6). Use the Law of Cosines on ΔPAC.

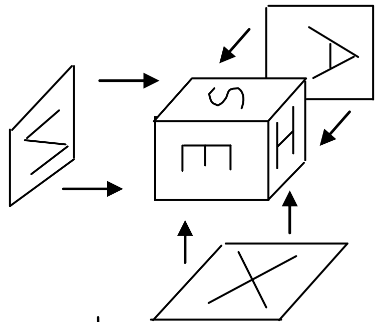


$$6^2 = 3^3 + 4^2 - 2(3)(4)\cos(\angle PCA) \Rightarrow \cos(\angle PCA) = -\frac{11}{24}$$

Therefore, $\cos(\angle BCA) = -\cos(\angle PCA) = \frac{11}{24}$. Use the Law of Cosines on ΔCAB. $AB^2 = 3^3 + 4^2 - 2(3)(4)\frac{11}{24} = 14$. Therefore, $AB = \sqrt{\mathbf{14}}$.

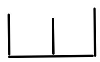
21. B Determine the unseen letters and their orientation by examining the second and third cubes.

From 2nd cube: broad side of N adjacent to extremity of S.

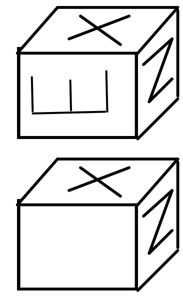


From 2nd cube: broad side of A adjacent to extremity of S.

From 3rd cube: broad side of X adjacent to extremity of H.

Answer is 

Rotating the cube around E and A until X is on top gives us the final view:



22. E Let Q(x) = P(x) - 10x. Then Q(1) = P(1) - 10 = 0, Q(2) = P(2) - 20 = 0, and Q(3) = P(3) - 30 = 0. Therefore, x = 1, 2, and 3 are roots of Q(x) = 0, and Q(X) = (x-1)(x-2)(x-3)(x-r) = 0.
Hence P(x) = (x-1)(x-2)(x-3)(x-r) + 10x. Now compute P(10) + P(-6)
P(10) + P(-6) = (9)(8)(7)(10-r) + 100 + (-7)(-8)(-9)(-6-r) - 60 = (9)(8)(7)[(10-r) - (-6-r)] + 40 = (9)(8)(7)(16) + 40 = **8104**.

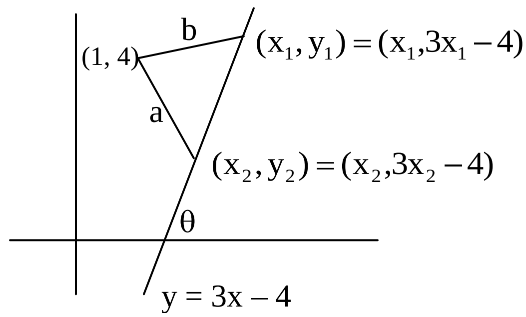
23. C Let p, b, and c represent the number of poodles, beagles, and collies, respectively. Then $\frac{1}{6}c \geq p \geq \frac{1}{5}b$ and $p + b \geq 23$. Since $p \geq \frac{1}{5}b$, $5p \geq b$, and $6p \geq b + p \geq 23$, therefore $6p \geq 23$ and $p \geq \frac{23}{6}$. But p must be a natural number, so $p \geq 4$. Since $\frac{1}{6}c \geq p \geq 4$, we have $\frac{1}{6}c \geq 4$ and $c \geq 24$. The values c = 24, p = 4, and b = 20 are consistent with the given information. Therefore, the minimum number of collies is **24**.

24. D Let a and b be the entries in two of the squares as shown and compute the two neighboring entries in terms of a and b. Then the common difference in the third row is b–2a, and in the fourth row it is 2b–a–74. Therefore, 2a + 4(b–2a) = 186 and a + 2(2b–a–74) =103. Solving these two equations gives a = 13 and b = 66. Then the entries in the third and fourth rows, and then in the fourth column can be completed, so that the ∗ represents 142. (The completed chart is shown below.)

			∗	
	74			
2a	b			186
a	2b–74	103		
0				

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

25. A Let θ be the measure of the angle the given line makes with the positive x-axis. Since the slope of the given line is 3, $\tan\theta = 3$. The other two sides of the triangle (a and b in the diagram) form angles with the positive x-axis whose measures are $\theta+60$ and $\theta+120$ degrees, respectively. Therefore, using the slope formula on a, and the formula for tan(A+B), we obtain



(i) $$\frac{3x_2 - 4 - 4}{x_2 - 1} = \tan(\theta + 60) = \frac{\tan\theta + \tan 60}{1 - \tan\theta\tan 60} = \frac{3+\sqrt{3}}{1-3\sqrt{3}}$$

Similarly, working with b, and noting that $\tan 120 = -\sqrt{3}$, we obtain

(ii) $$\frac{3x_1 - 4 - 4}{x_1 - 1} = \tan(\theta + 120) = \frac{\tan\theta + \tan 120}{1 - \tan\theta\tan 120} = \frac{3-\sqrt{3}}{1+3\sqrt{3}}.$$

Solving (i) we obtain $x_2 = \dfrac{15-\sqrt{3}}{6}$. Solving (ii) we obtain $x_1 = \dfrac{15+\sqrt{3}}{6}$.
Substituting these two values into the equation y= 3x – 4 and adding the two results, we obtain $y_1 + y_2 = 7$.