THE 2007-2008 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART II

February 6, 2008
Exam Time – 2 Hours

NO CALCULATORS

1. A set of N integers has the following property.

   If any one of the integers in the set is subtracted from the product of the others, the difference is divisible by N.

   For example, the set {4, 5, 8} has this property since $4\cdot5 - 8$, $4\cdot8 - 5$, and $5\cdot8 - 4$ are all divisible by 3. The set {3, 5, 7, 9} also has the property. Prove that in any such set, the sum of the squares of all the numbers is divisible by N.

2. Let P be a polynomial with integer coefficients and let a, b, c be integers. Suppose $P(a) = b$, $P(b) = c$, and $P(c) = a$. Prove that $a = b = c$.

3. Three points are chosen randomly, one on each side of triangle ABC (the chosen points are not A, B, or C). Prove that the circles determined by each vertex of the triangle and the two points chosen on the adjacent sides pass through a common point.

4. Determine all three digit positive integers N having the property that

   (i) N is divisible by 11 **and**

   (ii) $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

   Prove that you have found them all.

5. An oblique triangle is a triangle with no right angles. Given an oblique triangle ABC in which $\tan B = \tan^2 A$ and $\tan C = \tan^3 A$, compute, with proof, the value of $\tan C - \tan A$.

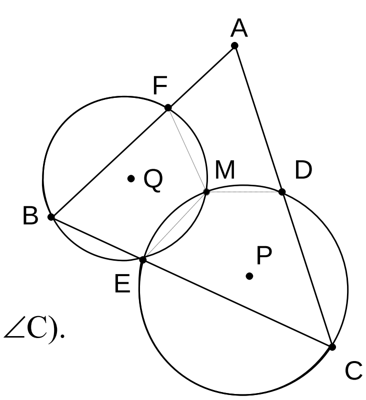
SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2007-08

1. Let the numbers be $a_1, a_2, a_3, \ldots, a_N$ and let P be their product. Then for each i, $1 \le i \le N$, N divides $\frac{P}{a_i} - a_i$. Therefore, N divides $a_i\left(a_i - \frac{P}{a_i}\right) = a_i^2 - P$.
It follows that N divides
$$a_1^2 - P + a_2^2 - P + \ldots + a_N^2 - P = (a_1^2 + a_2^2 + \ldots + a_N^2) - NP.$$
Thus, N divides $(a_1^2 + a_2^2 + \ldots + a_N^2)$.

2. Certainly, if two are equal (say $a = b$), then all three must be equal, since $P(b) = c \Rightarrow P(a) = c$, and we are also given $P(a) = b$. Thus $a = b = c$.
Suppose $a \ne b \ne c$. Without loss of generality assume that c is between a and b (i.e. $a < c < b$). Let $n_i x^i$ be a typical term of P. Then $n_i b^i - n_i a^i = n_i(b^i - a^i)$ is a term of $P(b) - P(a)$, and it has $b - a$ as a factor. But,
$$|P(b) - P(a)| = |c - b| < |b - a|.$$
This is a contradiction. Therefore, $a = b = c$.

Note: If the coefficients of P are not integers, the result no longer holds. For example, let $P(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2$. Then for a = 1, b = 2, c = 3 we have $P(1) = 2$, $P(2) = 3$, $P(3) = 1$.

3. Points D, E, and F are the randomly chosen points on sides $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$, respectively. Let circles Q and P, determined by points B, F, E, and C, D, E, respectively, intersect at M. Draw $\overline{FM}$, $\overline{ME}$, and $\overline{MD}$. Since the opposite angles of an inscribed quadrilateral are supplementary, $\angle FME = 180 - \angle B$ and $\angle DME = 180 - \angle C$. Therefore, $\angle FME + \angle DME = 360 - (\angle B + \angle C)$.
Since $\angle FMD = 360 - (\angle FME + \angle DME)$, we have
$$\angle FMD = \angle B + \angle C = 180 - \angle A.$$
Thus, $\angle FMD$ and $\angle A$ are supplementary, which proves quadrilateral AFMD is cyclic. Therefore, the circle determined by points A, F, and D also passes through point M.



4. Let $N = 100h + 10t + u$. Then by condition (ii) $N = 11(h^2 + t^2 + u^2)$.
Noting that $N = 100h + 10t + u = (99h + 11t) + (h - t + u)$, condition (i) will be only be satisfied if $(h - t + u)$ is divisible by 11. Since h, t, u are all ≤ 9, $(h - t + u)$ must equal 11 or 0.

Case 1: $h - t + u = 0 \Rightarrow t = h + u$. Therefore,

(1) $N = (99h + 11t) + (h - t + u) = 11(h^2 + t^2 + u^2)$

becomes $9h + (h + u) = h^2 + (h + u)^2 + u^2 \Rightarrow 10h + u = 2(h^2 + uh + u^2)$.

Therefore, u must be even. Rewriting the last equation as a quadratic in h,
$2h^2 + (2u - 10)h + 2u^2 - u = 0$. Since h is an integer, the discriminant of this equation must be a perfect square.

$(2u - 10)^2 - 8(2u^2 - u) = 4(25 - 8u - 3u^2)$ is only a perfect square when $u = 0$.

Hence, our quadratic equation becomes $2h^2 - 10h = 0$, so $h = 5$, and $t = h + u = 5$ and **N = 550**.

Case 2: $h - t + u = 11 \Rightarrow t = h + u - 11$. Therefore (1) becomes

$9h + (h + u - 11) + 1 = h^2 + (h + u - 11)^2 + u^2 \Rightarrow$

(2) $10h + u - 10 = 2[h^2 + uh + u^2 - 11(h + u)] + 121$

which means u must be odd. Rewriting (2) as a quadratic in h,
$2h^2 + (2u - 32)h + 2u^2 - 23u + 131 = 0$ whose discriminant $4(-3u^2 + 14u - 6)$ must be a perfect square. This only happens when $u = 3$. Substituting, the quadratic becomes $2h^2 - 26h + 80 = 0$ whose solutions are $h = 5$ and $h = 8$.

When $h = 5$, $t = 5 + 3 - 11 = -3$, which is unacceptable. When $h = 8$, $t = 8 + 3 - 11 = 0$, and **N = 803**.

Therefore, the only two solutions are **550** and **803**.

5. First we prove the following lemma:

In oblique triangle ABC, the sum of the tangents of the three angles is equal to the product of the tangents of the three angles [i.e. In oblique ΔABC, tan A + tan B + tan C = (tan A)(tan B)(tan C)].

Proof of lemma: Since B = 180 – (A + C), tan B = tan [180 – (A + B)] = – tan (A + C)
Using the well known sum formula,

tan B = – tan (A + C) = $-\dfrac{\tan A + \tan C}{1-(\tan A)(\tan C)}$. Clearing fractions we obtain,

tan B – (tan A)(tan B)(tan C) = tan A + tan C, and the lemma follows.

We are given that $\tan B = \tan^2 A$ and $\tan C = \tan^3 A$. Multiplying these two equations, and then multiplying the resulting equation by tan A, we obtain $(\tan A)(\tan B)(\tan C) = \tan^6 A$ or, using the lemma, $\tan^6 A = \tan A + \tan B + \tan C$.

Substituting for tan B and tan C, we obtain $\tan^6 A = \tan A + \tan^2 A + \tan^3 A$ and since tan A $\neq$ 0, $\tan^5 A - \tan A - \tan^2 A - 1 = 0$.

The left side of this last equation may be factored so that the equation becomes:

$$\tan A(\tan^4 A - 1) - (\tan^2 A + 1) = \tan A(\tan^2 A - 1)(\tan^2 A + 1) - (\tan^2 A + 1) =$$
$$(\tan^2 A + 1)(\tan^3 A - \tan A - 1) = 0.$$

Since $\tan^2 A + 1 \neq 0$, we have $\tan^3 A - \tan A - 1 = 0$ and $\tan^3 A - \tan A = 1$.

But we are given that $\tan C = \tan^3 A$. Substituting, tan C – tan A = **1**.