THE 2008–2009 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

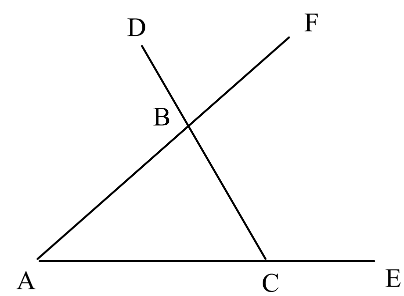
NO CALCULATORS

90 MINUTES

1. A boys and girls club found it could achieve a membership ratio of 2 girls to one boy either by inducting 24 girls or by expelling x boys. What is x?

   (A) 6 (B) 12 (C) 18 (D) 24 (E) 36

2. In the figure, the measure of ∠DBA is three times the measure of ∠DCA, and the measure of ∠ECD is twice the measure of ∠FAC. What is the degree measure of ∠ABC?



   (A) 36 (B) 45 (C) 50 (D) 72 (E) 75

3. For x ≠ y, if $\frac{3}{x} + 7y = \frac{3}{y} + 7x$, compute the product xy.

   (A) –3 (B) 7 (C) –21 (D) $-\frac{3}{7}$ (E) $\frac{7}{3}$

4. In the sequence 2007, a, b, 2008, … each term starting with the third term, b, equals the sum of the two previous terms. Compute the ratio of b to a.

   (A) $\frac{2008}{2007}$ (B) 2009 (C) $\frac{4015}{2}$ (D) $\frac{4015}{2008}$ (E) 4015

5. A *prime-prime* is a prime number that yields a prime when its units digit is omitted. (For example, 317 is a three-digit prime-prime because 317 is prime and 31 is prime). How many two-digit prime-primes are there? (Recall that 1 is not a prime number.)

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

6. Alan earns \$4 more in 4 hours than Joy earns in 3 hours. Joy earns \$0.50 more in 4 hours than Alan earns in 5 hours. How much does Alan earn per hour?

(A) \$15.00 (B) \$16.25 (C) \$17.50 (D) \$18.75 (E) \$19.00

7. Find the value of x which satisfies $\log_2(x+2) = 3 + \log_2 x$.

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{2}{7}$ (D) $\frac{3}{8}$ (E) $\frac{4}{9}$

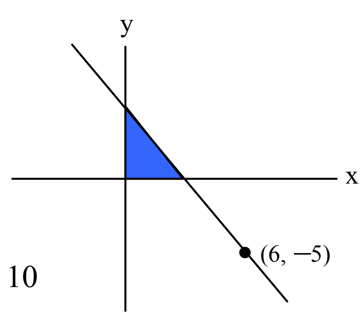
8. A driver wishes to arrive at her destination at exactly 11:00 a.m. If she drives at 30 mph, she would get there at 10:00 a.m. If she drives at 20 mph, she would arrive at noon. How fast should she drive to arrive at 11:00 a.m.

(A) 24 mph (B) 24.5 mph (C) 25 mph (D) 25.5 mph (E) 26 mph

9. The six basic trigonometric functions are sine, cosine, tangent, cosecant, secant, and cotangent. Compute the probability that a randomly chosen basic trigonometric function, when divided by a <u>different</u> randomly chosen basic trigonometric function, will result in a quotient that is itself one of the six basic trigonometric functions.

(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{7}{15}$ (D) $\frac{1}{2}$ (E) None of these

10. On the graph shown, a line passes through the point (6, –5) and intersects the positive x- and y-axes as shown. The shaded triangle has an area of 20. What is the y-intercept of the line?



(A) 8 (B) 8.5 (C) 9 (D) 9.5 (E) 10

11. 124 cartons, each containing the same number of marbles, have their contents emptied and repacked into 312 smaller boxes, each of which receives an equal number of marbles. What is the smallest possible (non-zero) total number of marbles involved?

(A) 2418 (B) 3224 (C) 4836 (D) 7254 (E) 9672

12. For all real numbers x, the function f(x) satisfies 2f(1+x) + 3f(1–x) = 5x. What is the value of f(2)?

(A) –2 (B) –3 (C) –4 (D) –5 (E) –6

13. A right triangle has sides of length 12, 16, and 20. A point chosen on the shortest side of the triangle is equidistant from the other two sides of the triangle. Compute this distance.

(A) 3 (B) $4\frac{1}{2}$ (C) 5 (D) $5\frac{1}{3}$ (E) 6

14. Suppose N is a positive integer such that $\frac{N}{52} < 1$. Compute the number of values of N for which $\frac{N}{52}$ is a non-reducible fraction.

(A) 23 (B) 24 (C) 25 (D) 26 (E) 27

15. One of the three integer roots of the equation $x^3 - ax^2 + bx + 4a = 0$ $(a \neq 0)$ is the negative of a second root. Compute the value of b.

(A) –4 (B) –2 (C) 0 (D) 2 (E) 4

16. Consider all sets of nickels, dimes, and quarters having at least one coin of each type and which add up to one dollar. What is the probability that a set of this kind will consist of a prime number of coins?

(A) $\frac{4}{13}$ (B) $\frac{1}{3}$ (C) $\frac{5}{13}$ (D) $\frac{5}{12}$ (E) $\frac{3}{7}$

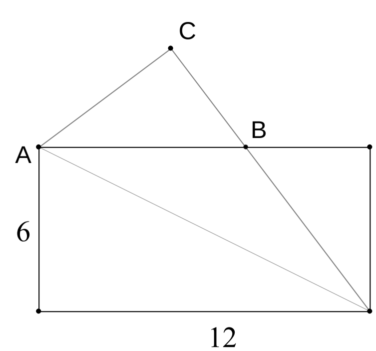
17. Compute the area of the region bounded by the graphs of $y = 4 - |x|$ and $y = |x| - 4$.

(A) 16 (B) $16\sqrt{2}$ (C) 24 (D) 32 (E) $32\sqrt{2}$

18. The sum of the first ten terms of an arithmetic sequence is four times the sum of the first five terms. What is the ratio of the second term to the first. (Note: all terms of the sequence are non-zero.)

(A) 2:1 (B) 3:1 (C) 4:1 (D) 5:1 (E) 6:1

19. A rectangular piece of paper with dimensions 6 by 12 is folded along one diagonal, as shown. What is the sine of angle BAC?



(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$ (E) $\frac{5}{6}$

20. The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 2008. What was the number of the page that was added twice?

(A) 55 (B) 64 (C) 72 (D) 75 (E) 82

21. In triangle ABC, sinA:sinB:sinC = 4:5:6, while cosA:cosB:cosC = x:y:2. Find the ordered pair (x, y).

(A) (2, 1) (B) (5, 4) (C) (9, 6) (D) (10, 5) (E) (12, 9)

22. Quadrilateral ABCD is inscribed in a circle. The degree measures of angles A, B, and C, in order, are integers that form an increasing geometric sequence. Compute the sum of all possible values for the measure of angle D.

(A) 126 (B) 180 (C) 234 (D) 272 (E) 360

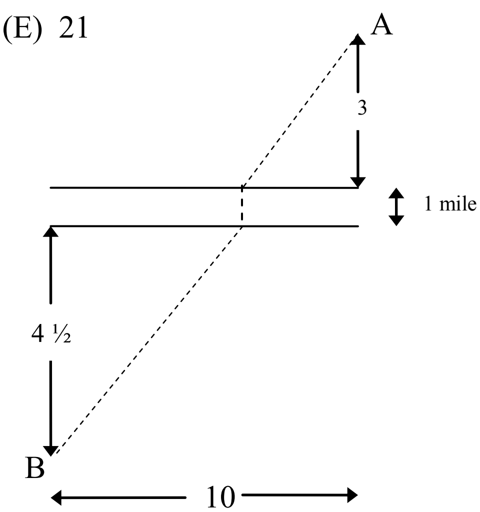
23. In a list of 200 numbers, every one (except the end ones) is equal to the sum of the two adjacent numbers in the list. The sum of all 200 numbers is equal to the sum of the first 100 of them. Find that sum if the thirty-sixth number in the list is 2008.

(A) 6024 (B) 4016 (C) 0 (D) –4016 (E) –6024

24. If *a*, *b* and *c* are three distinct numbers such that $a^2 - bc = 7$, $b^2 + ac = 7$, and $c^2 + ab = 7$, then compute $a^2 + b^2 + c^2$.

(A) 5 (B) 7 (C) 14 (D) 18 (E) 21

25. A straight canal is exactly 10 miles long and 1 mile wide. Point A is located 3 miles inland from one end of the canal and point B is located $4\frac{1}{2}$ miles inland from the other end of the canal on the opposite bank. A competitor starts from point A, runs to the canal, swims directly across the canal and then runs to point B. (His path is shown as the dotted segments.) Compute the least number of miles such a competitor may run.



(A) 10 (B) 10.5 (C) 12 (D) 12.5 (E) 13

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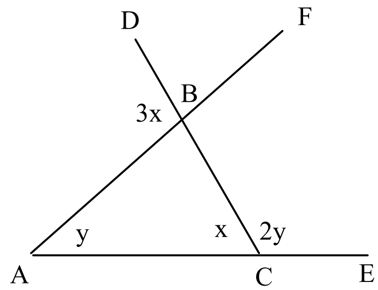
PART I - Solutions:

1. B If B = the number of boys and G = the number of girls, then G + 24 = 2B and G = 2(B–x). Thus, G + 24= G + 2x, so x=12.

2. D Represent the measures of the angles as shown. Then $3x = x + y$ and $2y = 180 - x$. From these two equations, $x = 36$ and $y = 72$ and $m\angle ABC = 180 - (36 + 72) = \mathbf{72}$.



3. D $\dfrac{3}{x} + 7y = \dfrac{3}{y} + 7x \Rightarrow 7y - 7x = \dfrac{3}{y} - \dfrac{3}{x} \Rightarrow 7(y - x) = \dfrac{3x - 3y}{xy} \Rightarrow xy = \dfrac{3(x - y)}{7(y - x)} = -\dfrac{\mathbf{3}}{\mathbf{7}}$

4. E $2007 + a = b$ and $2008 = a + b$. Solving gives $a = \dfrac{1}{2}$ and $b = \dfrac{4015}{2}$ so that $\dfrac{b}{a} = 4015$.

5. C For a two-digit number to be a prime-prime, its ten digit must be a prime. Thus only numbers in the 20s, 30s, 50s, and 70s are eligible. The primes in each of these groups are: 23, 29, 31, 37, 53, 59, 71, 73, 79, for a total of **9**.

6. C Let J = amount Joy earns in 1 hour and let A = amount Alan earns in 1 hour. Then 4A = 3J + 4 and 4J = 5A + .5. Solving these two equations together gives A = **$17.50**.

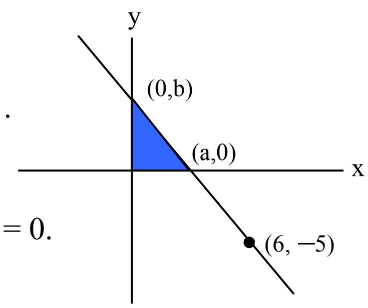
7. C $\log_2(x + 2) = 3 + \log_2 x \Rightarrow \log_2(x + 2) = \log_2 8 + \log_2 x \Rightarrow \log_2(x + 2) = \log_2(8x)$.
Therefore, $x + 2 = 8x$ and $x = \dfrac{\mathbf{2}}{\mathbf{7}}$.

8. A Let t represent the time it takes when she drives 30 mph. Then 30t = 20(t+2) and t = 4, so the distance is (30)(4) = 120 miles. Then in order to arrive at 11:00, she must travel for 5 hours at $\dfrac{120}{5} = \mathbf{24}$ mph.

9. B There are (6)(5) = 30 possible quotients. Using sine, cosine, and tangent in the numerator, the quotients that result in one of the basic functions are:

$$\frac{\sin}{\cos} = \tan, \frac{\sin}{\tan} = \cos, \frac{\cos}{\sin} = \tan, \frac{\cos}{\cot} = \sin, \frac{\tan}{\sin} = \csc, \frac{\tan}{\csc} = \sin .$$

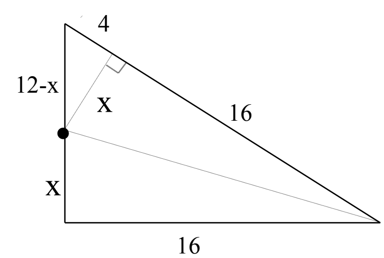
Each of these six equations above can be replaced by a corresponding equation in which all of the trig functions used are replaced by their reciprocal functions. Thus, there are a total of 12, and the required probability is $\frac{12}{30} = \mathbf{\frac{2}{5}}$.



10. E Area of the triangle = $\frac{1}{2}ab$ = 20, so ab = 40. Also, $\frac{b+5}{-6} = \frac{b}{-a}$.
Therefore, -ab – 5a = -6b $\Rightarrow$ ab = 6b – 5a or 40 = 6b – 5a.
Since a = $\frac{40}{b}$, we have 40 = 6b – 5($\frac{40}{b}$) and $6b^2 - 40b - 200 = 0$.
Dividing by 2 and factoring, (3b + 10)(b – 10) = 0, and b = **10**.

11. E 312 = $(2^3)(3)(13)$ and 124 = $(2^2)(31)$. The smallest possible total number of marbles is the LCM of the two numbers, $(2^3)(3)(13)(31)$ = **9672**.

12. D Using x = 1 in the given equation, 2f(1+1) + 3f(1–1) = 5(1) and 2f(2) + 3f(0) = 5
Using x = –1, 2f(1–1) + 3f(1+1) = 5(–1) and 2f(0) + 3f(2) = –5.
Solving the two equations for f(2) gives f(2) = **–5**.



13. D Construct a line segment from the point to the opposite vertex. The two triangles formed are congruent (HL). Thus the hypotenuse is divided into segments of 16 and 4. Using the Pythagorean Theorem,
$4^2 + x^2 = (12-x)^2$ from which x = $\mathbf{5\frac{1}{3}}$.

14. B There are 51 positive integers N < 52. Since 52 = $(2^2)(13)$, we must eliminate all those that are multiples of 2 or 13. There are 25 multiples of 2 (including 2·13 and 4·13). There are two odd multiples of 13 that are less than 52. Therefore, the total number of non-reducible fractions is 51 – (25 + 2) = **24**.

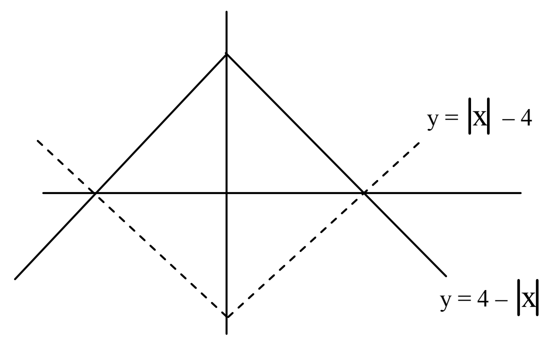
15. A In any equation of the form $x^3 + ax^2 + bx + c = 0$ with roots p, q, and r,
(i) p + q + r = -a, (ii) pq + pr + qr = b, and (iii) pqr = –c

Represent the roots of the given equation $x^3 - ax^2 + bx + 4a = 0$ by r, –r, and p.
Using (i) r + (–r) + p = a and p = a.
Using (iii) (r)(–r)(p) = (r)(–r)(a) = -4a. Therefore, $r^2 = 4$ and r = ± 2.
Thus, two of the roots of the given equation are 2 and -2.
Using (ii) (2)(–2) + (2)(p) + (–2)(p) = b and b = **–4**.

16. A Below is a list of all the combinations of nickels, dimes, and quarters that meet the requirements of the problem.

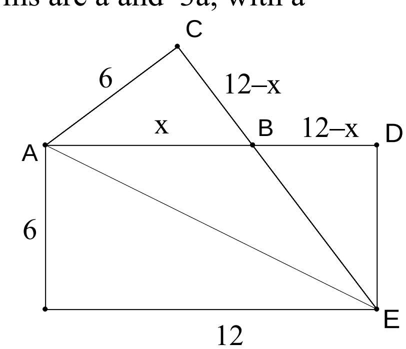
nickels	dimes	quarters	# of coins
1	7	1	9
3	6	1	10
5	5	1	11 prime
7	4	1	12
9	3	1	13 prime
11	2	1	14
13	1	1	15
2	4	2	8
4	3	2	9
6	2	2	10
8	1	2	11 prime
1	2	3	6
3	1	3	7 prime

Therefore, the desired probability is $\mathbf{\frac{4}{13}}$.



17. D The region is a square with sides of length $4\sqrt{2}$. The desired area is $(4\sqrt{2})^2 = \mathbf{32}$.

18. B Represent the sequence by a, a + d, a + 2d, . . ., a + 9d. Then the sum of the first 10 terms is 10a + 45d and the sum of the first 5 terms is 5a + 10d. Therefore, $10a + 45d = 4(5a + 10d) \Rightarrow d = 2a$. Hence, the first two terms are a and 3a, with a ratio of **3:1**.



19. B Clearly, AC = 6. Since $\Delta ABC \cong \Delta EBD$, represent the lengths as shown. Using the Pythagorean Theorem in right ΔABC, x = AB = 7.5 and BC = 4.5. Therefore, $\sin(\angle ABC) = \frac{4.5}{7.5} = \frac{3}{5}$.

20. A Let K be the number of the page that is added twice. Then $0 < K < n+1$ and $1 + 2 + 3 + . . . + n + K$ is between $1 + 2 + 3 + . . . + n$ and $1 + 2 + 3 + . . . + n + (n+1)$. Using the well known formula for the sum of the first n (or n+1) positive integers, this becomes $\frac{n(n+1)}{2} < 2008 < \frac{(n+1)(n+2)}{2}$ and $n(n + 1) < 4016 < (n + 1)(n + 2)$. Since $\sqrt{4016} \approx 63$, we find by inspection that n = 62. Therefore, $K = 2008 - \frac{(62)(63)}{2} = 55$.

21. E By the Law of Sines, the sides of triangle ABC are also in the ratio 4:5:6. Using the Law of Cosines,

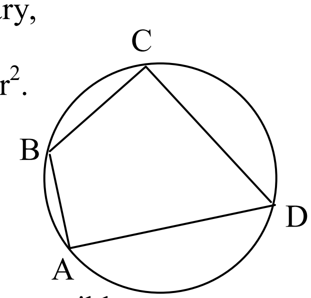
$$4^2 = 5^2 + 6^2 - 2(5)(6)\cos A \quad \Rightarrow \quad \cos A = \frac{3}{4} = \frac{12}{16}$$

$$5^2 = 4^2 + 6^2 - 2(4)(6)\cos B \quad \Rightarrow \quad \cos B = \frac{9}{16} \text{ and}$$

$$6^2 = 4^2 + 5^2 - 2(4)(5)\cos C \quad \Rightarrow \quad \cos C = \frac{1}{8} = \frac{2}{16}$$

Therefore, cosA:cosB:cosC = 12:9:2 and (x, y) = (12, 9).

22. C Since the opposite angles of an inscribed quadrilateral are supplementary, m<A + m<C = 180°. Since the measures of angles A, B, and C form a geometric sequence, represent the angles in order as a, ar, and ar^2. Then $a + ar^2 = a(1 + r^2) = 180 = 2^2 x 3^2 x 5$. Hence, $(1 + r^2)$ is a factor of 180. The only factors of 180 which are 1 more than a perfect square are $2 = 1^2 + 1$, $5 = 2^2 + 1$, and $10 = 3^2 + 1$. Now, $r \neq 1$ since the progression increases. If r = 2, then a = 36 and m<A = 36, m<B = 72, m<C = 144, and m<D = 108. If r = 3, then a = 18 and m<A = 18, m<B = 54, m<C = 162, and m<D = 126. Therefore, the possible values for the measure of <D are 108 and 126, and the desired sum is **234.**



23. E Let x_n represent the n^{th} number in the list. If $x_1 = a$, $x_2 = b$, then $x_3 = b - a$, $x_4 = -a$, $x_5 = -b$, $x_6 = a - b$, $x_7 = a$, $x_8 = b$, etc. Therefore, the sequence has period 6 (repeats itself every six terms). If S_n represents the sum of the first n terms, then $S_6 = 0$, $S_{100} = 2b - a$, $S_{200} = a + b$. Since $S_{100} = S_{200}$, then $2a - b = 0$. We are given $x_{36} = 2008$, so that $a - b = 2008$. Thus $a = -2008$, $b = -4016$, and $S_{200} = (-2008 + -4016) = $ **–6024**.

24. C Subtract the 2^{nd} and 3^{rd} equations to get $0 = b^2 - c^2 - a(b-c) = (b+c-a)(b-c)$. Since b and c are distinct, $b + c = a$. Add all three equations and regroup:

$21 = a^2 + b^2 + c^2 + ac + ab - bc = a^2 + b^2 + c^2 + a(b+c) - bc =$

$a^2 + b^2 + c^2 + a^2 - bc = a^2 + b^2 + c^2 + 7$. Thus, $a^2 + b^2 + c^2 =$ **<u>14</u>**.

25. D The two right triangles ACD and BEF are similar since $EF \| CD$ and $BE \| AC$. The ratio of their corresponding sides is $\dfrac{3}{4\frac{1}{2}} = \dfrac{2}{3}$. Therefore, $CD = \dfrac{2}{5}(10) = 4$ and $EF = \dfrac{3}{5}(10) = 6$. Using the Pythagorean Theorem on each triangle, AD = 5, BF = $7\frac{1}{2}$, and the shortest distance the competitor can run is **12.5** miles.