THE 2010–2011 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box, or more than one box, is marked. Zero points are given for an incorrect answer. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Although Elvis Presley had more top ten hits than the Beatles (Elvis had 38, the Beatles had 33), the Beatles had two more of these top ten hits reach number one than Elvis had. If between them there were a total of 33 songs that reached the top ten but <u>not</u> number one, how many number one hits did the Beatles have?

(A) 14 (B) 16 (C) 18 (D) 20 (E) 22

2. A date is called *weird* if the number of its month and the number of its day are relatively prime (have greatest common factor 1). For example, April 21 is *weird* because 4 and 21 are relatively prime. What is the smallest number of *weird* days in any month?

(A) 15 (B) 14 (C) 13 (D) 11 (E) 10

3. A “magic square” is a square matrix of numbers (not necessarily positive and not necessarily integers) in which all rows, all columns, and two diagonals add up to the same value. In the partially filled-in magic square to the right, each row, column, and diagonal <u>must</u> add up to what number?

2	3	
		4
		1

(A) 4.25 (B) 4.5 (C) 5 (D) 5.25 (E) 6.5

4. If L has equation ax + by = c, M is its reflection across the y-axis, and N is its reflection across the x-axis, which of the following must be true about M and N for all nonzero choices of a, b, and c?

(A) the x-intercepts are equal
(B) the y-intercepts are equal
(C) the slopes are equal
(D). the slopes are reciprocals
(E) the slopes are negative reciprocals

5. The counting numbers are written in the pattern at the right. What is the middle number of the 40^{th} row?

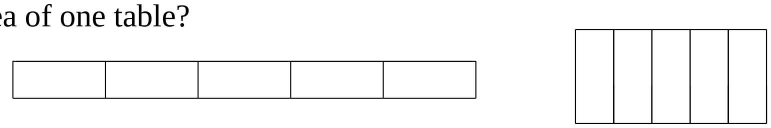
1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16

(A) 1561 (B) 1641 (C) 1559 (D) 1639 (E) 1483

6. Consider the quadratic equation $ax^2 + bx + c = 0$, for fixed values of b and c. When the value of a is 2, the equation has a root of 2, and when the value of a is 1, the equation has a root of 1. Compute the root of the equation when the value of a is 0.

(A) 0 (B) $\frac{2}{3}$ (C) $\frac{6}{7}$ (D) $\frac{3}{2}$ (E) $\frac{11}{6}$

7. When five identical tables are placed end-to-end as on the left, the ratio of perimeter to area of the resulting shape is 1/2; when they are placed side-by-side as on the right, the ratio of perimeter to area is 3/10. What is the ratio of perimeter to area of one table?



(A) 2/3 (B) 3/4 (C) 4/5 (D) 11/7 (E) 32/11

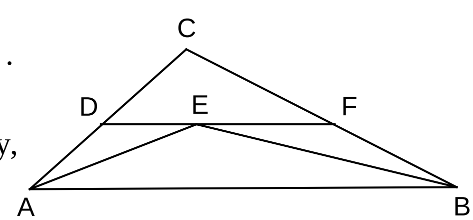
8. Chuck bought groceries with a $10 bill. The cost of the groceries had 3 different digits, and the amount of his change had the same 3 digits in a different order. What was the sum of the digits in the cost?

(A) 13 (B) 14 (C) 15 (D) 16 (E) Cannot be determined

9. If Tom gets 71 on his next quiz, his average will be 83. If he gets 99, his average will be 87. How many quizzes has Tom already taken (assume all quizzes are weighted equally)?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

10. In the diagram, ABC is a triangle with $\overline{DF}$ parallel to $\overline{AB}$. Also, $\overline{AE}$ bisects $\angle CAB$, and $\overline{BE}$ bisects $\angle CBA$. If the lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are 12, 8, and 6, respectively, what is the perimeter of triangle CDF?

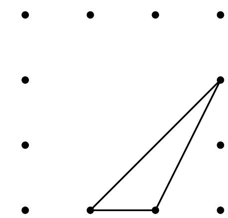


(A) 12 (B) 13 (C) 14 (D) 16 (E) 18

11. Don traveled 1 hour longer and 2 miles farther than Debbie, but averaged 3 mph slower. If the sum of their times was 4 hours, what was the sum in miles of the distances they traveled?

(A) 5 (B) 26 (C) 28.5 (D) 30.5 (E) 46

12. Twelve lattice points are arranged along the edges of a 3×3 square as shown. How many triangles have all three of their vertices among these points? One such triangle is shown.



(A) 48 (B) 64 (C) 204 (D) 220 (E) 256

13. The year 2010 has three consecutive prime number factors, 2, 3, and 5. How many years in the third millenium, i.e. between the years 2001 and 2999, have three consecutive prime number factors (including 2010)?

(A) 40 (B) 41 (C) 45 (D) 46 (E) 48

14. Compute the value of A which satisfies the equation $\frac{\log 3 + \log 4 + \log 5}{\log 6 + \log 8 + \log A} = \frac{1}{2}$.

(A) 10 (B) 20 (C) 45 (D) 60 (E) 75

15. A sheet of graph paper is folded once so that the point (-5, 3) folds onto the point (4,-3). At what point does the crease (the fold line) cross the y-axis?

(A) (0, 0) (B) $(-\frac{1}{2}, 0)$ (C) $(0, \frac{1}{2})$ (D) $(0, \frac{3}{4})$ (E) $(0, \frac{3}{2})$

16. Three children are all under the age of 15. If I tell you that the product of their ages is 90, you do not have enough information to determine their ages. If I also tell you the sum of their ages, you still do not have enough information to determine their ages. Which of the following is NOT a possible age for one of the children?

(A) 2 (B) 3 (C) 5 (D) 6 (E) 9

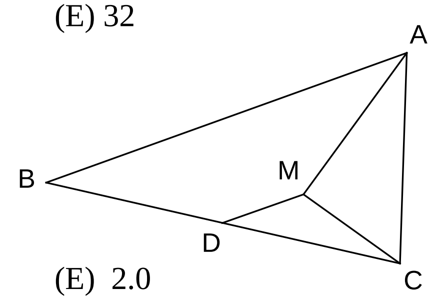
17. Evaluate $\cos^2(1°) + \cos^2(2°) + \cos^2(3°) + \ldots + \cos^2(90°)$.

(A) 40.5 (B) 42.25 (C) 42.5 (D) 44.25 (E) 44.5

18. In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. What is the sum of the first four terms?

(A) 13 (B) 14 (C) 26 (D) 28 (E) 32

19. In the diagram, AB = 6.8, BC = 5.5, AC = 4.6, $\overline{AM}$ bisects <BAC, $\overline{CM} \perp \overline{AM}$, and D is the midpoint of $\overline{BC}$. Compute the length of $\overline{DM}$.



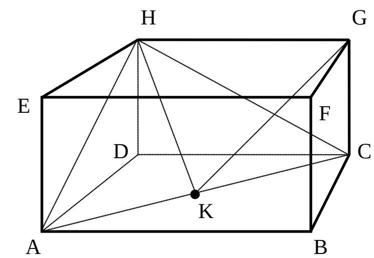
(A) 1.0 (B) 1.1 (C) 1.4 (D) 1.7 (E) 2.0

20. Suppose f(x) = ax + b, g(x) = bx + a (a, b integers). If f(1) = 8 and f(g(50)) - g(f(50)) = 28, find the product of a and b.

(A) 5 (B) 12 (C) 48 (D) 182 (E) 210

21. One of the roots of the equation $x^3 - 14x^2 + 26x + c = 0$ is the quotient of the other two roots. Compute the <u>product</u> of all possible values of c.

(A) 456 (B) 529 (C) 576 (D) 625 (E) 676



22. In the rectangular solid, the areas of the following rectangles are: EFGH = 300, BCGF = 240, ABFE = 320. If HK is perpendicular to AC, compute the area of triangle GHK.

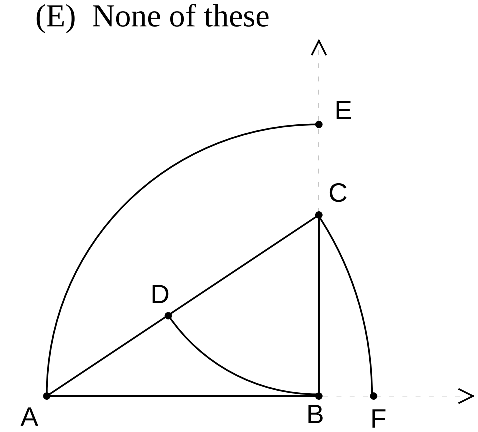
(A) $8\sqrt{17}$ (B) $16\sqrt{34}$ (C) $12\sqrt{51}$ (D) $32\sqrt{34}$ (E) $40\sqrt{17}$

23. Two boxes each contain both blue and green beads. Each box contains 6 blue beads, but each has a different number of green beads. If two beads are randomly drawn from one box, the probability of drawing one of each color is $\frac{1}{2}$. If two beads are randomly drawn from the other box, the probability of drawing one of each color is also $\frac{1}{2}$. What is the total number of green beads in the two boxes?

(A) 6 (B) 8 (C) 9 (D) 12 (E) 13

24. How many triples of positive integers (a, b, c) are there with a < b < c and a + b + c = 401?

(A) 13,200 (B) 15,750 (C) 17,200 (D) 19,700 (E) None of these



25. ABC is a right triangle with BC = 3, AB = 4, and AC = 5. The circle with center C and radius $\overline{CB}$ meets $\overline{AC}$ at D, the circle with center B and radius $\overline{BA}$ meets $\overline{BC}$ (extended) at E, and the circle with center A and radius $\overline{AC}$ meets $\overline{AB}$ (extended) at F. Compute the area of triangle DEF (not shown).

(A) 6 (B) $\frac{31}{5}$ (C) 6.5 (D) 7 (E) $\frac{46}{5}$

SOLUTIONS – KSU MATHEMATICS COMPETITION – 2010

1. D Let B = number of #1 hits for the Beatles, Let E = number of #1 hits for Elvis. Then B = E + 2 and 33 – B + 38 – E = 33. Solving gives E = 18 and B = **20**.

2. E Clearly, the number of the month in question must have the largest possible number of prime factors in order to minimize the number days with which it has a common factor other than 1. These are months 6, 10, and 12 (June, October, and December). Each has two prime factors, but October and December have 31 days, and 31 is a prime number. So let's consider June first. There are 10 days in June whose greatest common factor with 6 is 1 (1, 5, 7, 11, 13, 17, 19, 23, 25, 29). [There are 13 days in October whose greatest common factor with 10 is 1 (1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, and 31). There are 11 days in December whose greatest common factor with 12 is 1 (1, 5, 7, 11, 13, 17, 19, 23, 25, 29, and 31).] Therefore, the correct month is June and the number of "weird" days is 10.

3. B There are several ways to proceed. Here is one approach.

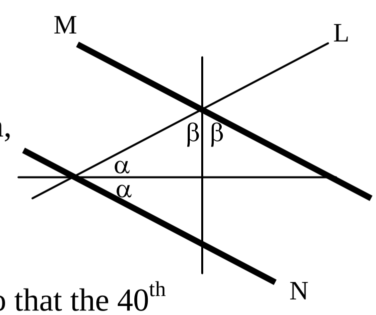
2	3	X
	B	4
A	C	1

The desired sum is X + 5.
Since 2+B+1 = X+5, B = X+2.
Since A+B+X = A + X + 2 + X = X + 5, A = 3 – X
Since 3 + B + C = 3 + X + 2 + C = X + 5, C = 0
Thus, the magic square becomes:

2	3	X
	X+2	4
3-X	0	1

Using the bottom row, 3 - X + 0 + 1 = X + 5 which yields X = -.5. Therefore, each row, column, and main diagonal has a sum of **4.5**.

4. C Even a quick rough sketch like the one shown demonstrates that the reflection images are parallel and thus have the same slope. Proof: Reflection preserves angle measurement. From the diagram, $\alpha + \beta = 90$, therefore $2\alpha + 2\beta = 180°$ so that the same-side interior angles of M and N are supplementary, proving the lines parallel.



5. A Note that the number on the right of each row is a perfect square, so that the 40th row ends in $40^2 = 1600$. Also note that the first number in the 40th row is $39^2 + 1 = 1522$. Since each row has an odd number of numbers, the middle number of the 40th row is the average of the first and last or 1561.

6. C When a = 0, $x = -\frac{c}{b}$. We know $2(2^2) + 2b + c = 0 \Rightarrow 2b + c = -8$. We also know $1 + b + c = 0 \Rightarrow b + c = -1$. Solving the two equations together, b = -7 and c = 6. Therefore, when a = 0, $x = \mathbf{\frac{6}{7}}$.

7. A Representing the dimensions of each individual rectangle as L and W and its area by A, we have $\frac{10L+2W}{5A}=\frac{1}{2}$ and $\frac{10W+2L}{5A}=\frac{3}{10}$. Adding these two equations, $\frac{12L+12W}{5A}=\frac{4}{5}$. The left numerator is 6 times the perimeter, P, of an individual rectangle. Therefore, $\frac{6P}{5A}=\frac{4}{5}$ or $\frac{P}{A}=\frac{2}{3}$.

8. B Since both the cost and the amount of change each have three digits, neither can be more than \$9. Thus the first digit of each must sum to 9. Also the last digit of each must be 5 or else four different digits would have to be used in the cost and the change. Thus, the sum of the digits in each must be 9 + 5 = 14. The possible costs that satisfy the conditions are \$8.15, \$7.25, \$6.35, \$3.65, \$2.75, and \$1.85.

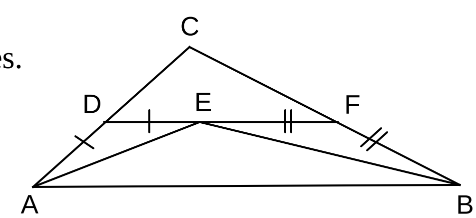
9. C Let n = the number of quizzes Tom has already taken, and let x = the sum of the scores on all the quizzes he has already taken. Then

$$\frac{x+71}{n+1}=83 \qquad \Rightarrow \qquad x=83n+12$$

$$\frac{x+99}{n+1}=87 \qquad \Rightarrow \qquad x=87n-12$$

Solving these two equations together, n = 6.

10. C $\angle EAB$ and $\angle DEA$ are congruent alternate interior angles. Therefore, $\angle EAB \cong \angle DEA \cong \angle DAE$, making ΔAED isosceles. Similarly, ΔBEF is isosceles. Thus, the perimeter of ΔDCF is equal to CA + CB = 6 + 8 = 14.



11. D Representing Debbie's time by x, we have from the given information, x + x + 1 = 4 from which x = $\frac{3}{2}$. Let's make a chart of the given information. Then $\frac{3}{2}y+2=\frac{5}{2}(y-3)$ from which y = $\frac{19}{2}$. Thus Don traveled $\frac{65}{4}=16.25$ miles and Debbie traveled $\frac{57}{4}=14.25$ miles, for a total of 30.5 miles.

	R	T	D
Don	y-3	5/2	(5/2)(y-3)
Debbie	y	3/2	(3/2)y

12. C Every set of three points except those which are collinear can be the vertices of a triangle. There are ${}_{12}C_3$ = 220 three element subsets, and $4({}_4C_3)$ = 16 which are collinear, so there are 220 − 16 = 204 triangles.

13. A Let us first list the sets of three consecutive primes whose product is less than 3000. They are:

$2 \times 3 \times 5 = 30$
$3 \times 5 \times 7 = 105$
$5 \times 7 \times 11 = 385$
$7 \times 11 \times 13 = 1001$
$11 \times 13 \times 17 = 2431$

We now need to count the number of multiples of these products which lie between 2001 and 2999. For the last 2 products above, there is only one multiple of each in the desired range, namely 2002 and 2431. For 385 we get only 2310 and 2695 in the desired range. For 105, on the other hand, we get nine multiples: 2100, 2205, 2310, 2415, 2520, 2625, 2730, 2835, and 2940. This gives a total of 13 multiples so far, except that the number 2310 has been accounted for twice in this list. Thus, we have so far found 12 such multiples. There are 33 multiples of 30 between 2001 and 2999, however this counts the five even multiples of 105 twice. Therefore, the total number of years satisfying the given condition is $12 + 33 - 5 = 40$.

14. E $\dfrac{\log 3+\log 4+\log 5}{\log 6+\log 8+\log A}=\dfrac{\log 60}{\log 48A}$. Therefore, $\dfrac{\log 60}{\log 48A}=\dfrac{1}{2} \Rightarrow$

$2\log 60 = \log 48A \Rightarrow \log 60^2 = \log 48A$. Thus $60^2 = 3600 = 48A$, and A = **75.**

15. D Let l be the line through (-5, 3) and (4, -3). The fold line is the perpendicular bisector of l , and so passes through midpoint $(-\frac{1}{2}, 0)$. Since the slope of l is $-\frac{2}{3}$, the slope of the fold line is $\frac{3}{2}$. Thus, the equation of the fold line is $y = \frac{3}{2}x + \frac{3}{4}$, and the point at which it crosses the Y-axis is $\mathbf{(0, \frac{3}{4})}$.

16. D Let a, b, c be the ages of the three children, with $a \le b \le c < 15$. Since we know the product $abc = 90$, each is a factor of 90 and each lies between 1 and 14 (inclusive). The only possibilities for a, b, and c are 1, 2, 3, 5, 6, 9, and 10. Consider the possible products which satisfy these conditions and for each compute the sum of the ages.

a	b	c	sum
1	9	10	20
2	5	9	16
3	3	10	16
3	5	6	14

We are also told that even knowing the sum of the ages would not allow us to determine the ages. Thus, the sum must be 16, since there are two different sets of ages that sum to 16. The only ages found in these two sets are 2, 3, 5, 9, and 10. Ages 1 and 6 do not appear.

17. E Let S denote the given sum. Since $\cos(90°) = 0$, $\cos(x) = \sin(90° - x)$ and

$\sin^2(x) = 1 - \cos^2(x)$, we get $S = \cos^2(1°) + \cos^2(2°) + \cos^2(3°) + \ldots + \cos^2(90°)$

$= \sin^2(90° - 1°) + \sin^2(90° - 2°) + \sin^2(90° - 3°) + \ldots + \sin^2(90° - 89°)$

$= \sin^2(89°) + \sin^2(88°) + \sin^2(87°) + \ldots + \sin^2(1°)$

$= (1 - \cos^2(89°)) + (1 - \cos^2(88°)) + (1 - \cos^2(87°)) + \ldots + (1 - \cos^2(1°)) = 89 - S$

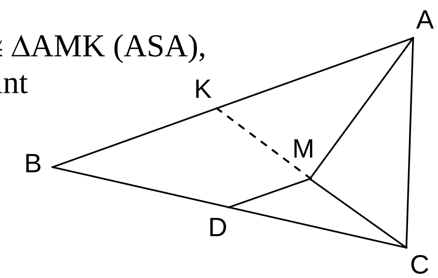
Thus $2S = 89$, which implies that $S = 44.5$.

18. D Represent the first six terms of the geometric sequence as $a, ar, ar^2, ar^3, ar^4, ar^5$.
Then $a + ar + ar^2 + ar^3 + ar^4 + ar^5 = 91$ and $a + ar = 7$.
Grouping the left side of the first equation in pairs, and factoring:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 = (a + ar)(1 + r^2 + r^4).$$

Thus, $7(1 + r^2 + r^4) = 91$ and $1 + r^2 + r^4 = 13$. Rewriting the last equation as $r^4 + r^2 - 12 = 0$ and factoring, $(r^2 + 4)(r^2 - 3) = 0$, and $r^2 = 3$. Thus, the sum of the first four terms is $a + ar + ar^2 + ar^3 = (a + ar)(1 + r^2) = (7)(4) = \mathbf{28}$.

19. B Extend $\overline{CM}$ through M to meet $\overline{AB}$ at K. Since $\Delta AMC \cong \Delta AMK$ (ASA), $AK = AC = 4.6$, making $BK = 2.2$. Since M is the midpoint of $\overline{KC}$, $\overline{MD}$ connects the is the midpoints of two sides of ΔBKC. Therefore, $MD = \frac{1}{2}(BK) = \mathbf{1.1}$



20. B $f(g(x)) - g(f(x)) = a^2 + b - b^2 - a$. Since $f(1) = a + b = 8$, $b = 8 - a$. Substituting, We obtain $a^2 + 8 - a - (8 - a)^2 - a = 14a - 56 = 28$ from which $a = 6$, $b = 2$ and $ab = \mathbf{12}$.

21. E In any equation of the form $x^3 + ax^2 + bx + c = 0$ with roots p, q, and r,

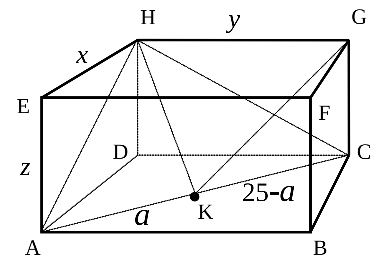
(i) $p + q + r = -a$, (ii) $pq + pr + qr = b$, and (iii) $pqr = -c$

Represent the roots of the given equation $x^3 - 14x^2 + 26x + c = 0$ by p, q, and $\frac{p}{q}$.

Using (i) $p + q + \frac{p}{q} = 14 \Rightarrow q^2 + p = 14q - pq$.

Using (ii) $(p)(q) + (p)(\frac{p}{q}) + (q)(\frac{p}{q}) = 26 \Rightarrow p(q^2 + p + q) = 26q$. Substituting $14q - pq$ for $q^2 + p$ in this last expression, we obtain $p(14q - pq + q) = 26q \Rightarrow$ $15pq - p^2q = 26q \Rightarrow p^2 - 15p + 26 = 0$ from which $p = 2, 13$. Substituting each of these into the given equation, we obtain $c = -4$ and $c = -169$. The product of these two values is 676.

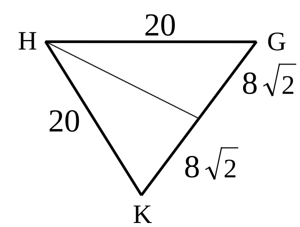
22. D Solving the equations $xy = 300$, $xz = 240$ and $yz = 320$ simultaneously gives the dimensions of the rectangular solid: $x = 15$, $y = 20$, and $z = 16$.



Using the Pythagorean Theorem on right triangles HEA, HGC, and ABC we find HA = $\sqrt{481}$, HC = $\sqrt{656}$, and AC = 25.

Representing HK as p, AK as a and KC as $(25-a)$, and using the Pythagorean Theorem on right triangles HAK and HKC,

$$a^2 + p^2 = 481 \quad \text{and} \quad (25-a)^2 + p^2 = 656$$



Solving these two equations together gives $a = 9$ and $p = 20$. Noting that triangle GCK is an isosceles right triangle with legs of 16, GK = $16\sqrt{2}$.

Since HG = HK = 20, ΔGHK is isosceles. Its area can be found by drawing an altitude to GK (length $\sqrt{272}$) and using area = ½bh. The area of ΔGHK is $32\sqrt{34}$.

23. E Let g be the number of green beads in a box. Then the total number of beads in that box is g + 6. The number of ways of drawing two beads together from the box is ${}_{(g+6)}C_2 = \dfrac{(g+6)(g+5)}{2}$. The number of ways of drawing together 1 green and 1 blue bead is 6·g. Thus, the probability of drawing 1 of each color is

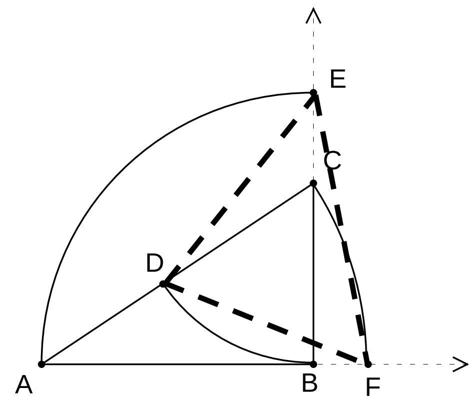
$$\frac{6g}{\frac{(g+6)(g+5)}{2}} = \frac{12g}{(g+6)(g+5)} = \frac{1}{2}$$

Solving this equation gives two values of g, g = 3 and g = 10. Therefore, the total number of green beads is 13.

24. A Examining the possible triples starting with the largest values of a, a pattern emerges. There are two triples beginning with 132 [(132, 134, 135) and (132, 133, 136)]. There are three triples beginning with 131 [(131, 134, 136), (131, 133, 137), and (131, 132, 138)]. There are five triples beginning with 130 [(130, 135, 136), (130, 134, 137), (130, 133, 138), (130, 132, 139), and (130, 131, 140)]. Continuing in this way creates a sequence of consecutive integers beginning with 2 and missing every 3rd number: 2, 3, 5, 6, 8, 9,…, 195, 197, 198. The missing numbers are at 4, 7, 10, …, 196. Therefore, we need to compute the value of the sum

$$(2 + 3 + 4 + \ldots + 198) - (4 + 7 + 10 + \ldots + 196) = \frac{200(197)}{2} - \frac{200(65)}{2} = 13{,}200.$$

25. B The following lengths can be determined fairly easily:
EC = 1, CD = 3, AD = 2, BF = 1, EB = 4, AF = 5.



Also using right triangle ABC, $\sin\angle BCA = \frac{4}{5}$ and

$\sin\angle DAB = \frac{3}{5}$. The area of ΔDEF is equal to

area ΔABC + area ΔEBF + area ΔEDC - area ΔADF

Now: area $\Delta ABC = \frac{1}{2}(3)(4) = 6$

area $\Delta EBF = \frac{1}{2}(EB)(BF) = \frac{1}{2}(4)(1) = 2$

area $\Delta EDC = \frac{1}{2}(EC)(CD)(\sin\angle ECD) = \frac{1}{2}(EC)(CD)(\sin\angle BCA) = \frac{1}{2}(3)(1)(\frac{4}{5}) = \frac{6}{5}$

area $\Delta ADF = \frac{1}{2}(AD)(AF)(\sin\angle DAB) = \frac{1}{2}(2)(5)(\frac{3}{5}) = 3$

Therefore, the area of $\Delta DEF = 6 + 2 + \frac{6}{5} - 3 = \mathbf{\frac{31}{5}}$.