THE 2010-2011 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

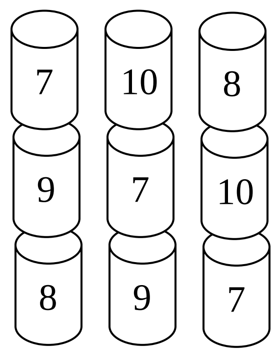
Calculators are NOT permitted

1. For each real number A, let M represent the minimum value of the numbers $5A + 1$, $A + 2$, and $-2A + 6$. For example, if $A = 1$, the three numbers are 6, 3, and 4. In this case, $M = 3$. Determine, with proof, the maximum value of M.

2. Numbers have the same parity if both are even or both are odd. Suppose a and b are integers greater than 1. Prove that if a and b have different parity, then $\log_b a$ is irrational. Prove that the converse is NOT true.

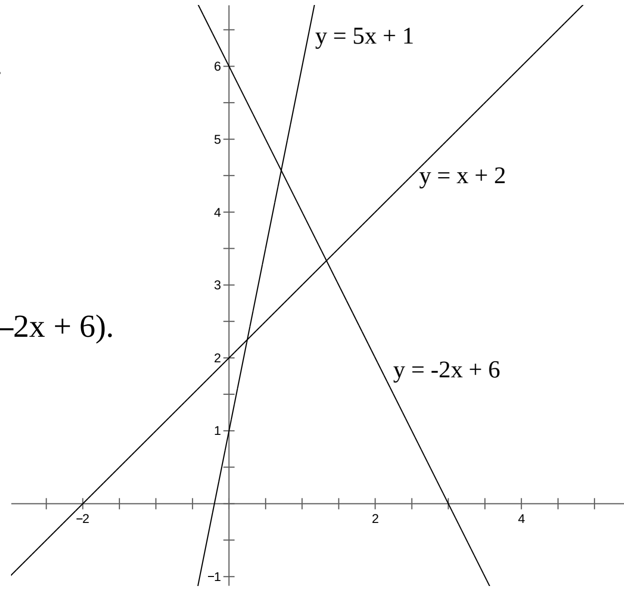
3. Find all solutions (x, y) of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{2011}$, where x and y are integers, and prove that you have found them all.

4. At a carnival game, you see nine paint cans stacked and numbered as shown at the right. You get three throws, and you must knock down one, and only one, can per throw. Further, a can may only be knocked down after the one(s) directly above it have been knocked down on a previous throw. Your first throw scores the number on that can, the second throw scores twice the number on that can, and the third throw scores triple the number on that can. To win a prize, you must score exactly 50 points. Determine, with proof, the number of possible combinations of three throws that can win a prize.



5. The lengths of the sides of a parallelogram are 9 inches and 8 inches. The lengths of the two diagonals, d_1 and d_2, of this parallelogram are both integers with $d_1 < d_2$. Compute, with proof, all possible ordered pairs (d_1, d_2).

SOLUTIONS



1. Make a graph y = 5x + 1, y = x + 2, and y = –2x + 6. The points of intersection of each pair of lines (from left to right) are

$$\left(\frac{1}{4},\ 2\frac{1}{4}\right),\ \left(\frac{5}{7},\ 4\frac{4}{7}\right),\ \text{and}\ \left(\frac{4}{3},\ 3\frac{1}{3}\right).$$

Consider the intersection furthest to the right (i.e. $\left(\frac{4}{3},\ 3\frac{1}{3}\right)$, the intersection of y = x + 2 and y = –2x + 6). At any other point, at least one of the three lines is below this intersection point. Therefore, the value of M at any other point is less than the value of M at the point $\left(\frac{4}{3},\ 3\frac{1}{3}\right)$. Therefore, the maximum value of M is $\frac{10}{3} = \mathbf{3\frac{1}{3}}$.

2. Using the change of base formula, $\log_b a = \frac{\log a}{\log b}$. Suppose $\log_b a$ is rational.

Then $\log_b a = \frac{\log a}{\log b} = \frac{h}{k}$, where h and k are integers and $k \neq 0$ and $h \neq 0$ (since a > 1).

Then $k \log a = h \log b$. Thus, $\log a^k = \log b^h$, which implies $a^k = b^h$.

But since h, $k \neq 0$, and a and b have different parity, a^k and b^h have different parity.

We have a contradiction. Therefore, $\log_b a$ is irrational.

The converse states that if $\log_b a$ is irrational, then a and b have different parity.

This is equivalent to: If a and b have the same parity, then $\log_b a$ is rational.

Here is a counterexample:

Consider $\log_2 6$ (note that 2 and 6 have the same parity). We must show that $\log_2 6$ is irrational. Assume $\log_2 6$ is rational. Then $\log_2 6 = \frac{p}{q}$ where p and q are integers and $q \neq 0$.

Then $2^{\frac{p}{q}} = 6$ or $2^p = 6^q = (2 \cdot 3)^q = (2^q)(3^q)$. This can only happen if p = q = 0 which is a contradiction. Therefore, the converse is not true.

3. The given equation implies that $xy = 2011(x+y)$. Since 2011 is prime, to have integer solutions at least one of x or y must be divisible by 2011. Since the equation is symmetric we can assume $x = 2011k$ where k is an integer. Then

$$xy = 2011(x + y) \Rightarrow ky = 2011k + y.$$

From the last equality we see that $2011k + y$ must be divisible by k. Since $2011k$ is divisible by k, $y = km$ where m is an integer. Therefore,

$$ky = 2011k + y \quad \Rightarrow \quad km = 2011 + m \quad \Rightarrow \quad m(k - 1) = 2011.$$

Since m and k are both integers, and 2011 is a prime number, there are only 4 possible combinations for which the above equation holds:

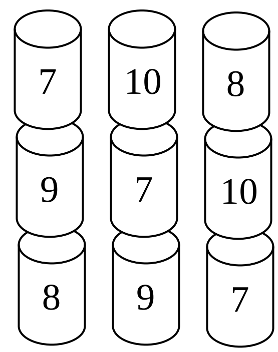
$m = 2011;\ k - 1 = 1 \quad \Rightarrow \quad x = 4022;\ y = 4022$

$m = -2011;\ k - 1 = -1 \quad \Rightarrow \quad x = 0;\ y = 0$ (reject)

$m = 1;\ k - 1 = 2011 \quad \Rightarrow \quad x = (2011)(2012) = 4046132;\ y = 2012$

$m = -1;\ k - 1 = -2011 \quad \Rightarrow \quad x = (-2010)(2011) = -4042110;\ y = 2010:$

Hence we have 5 possible solutions $(x;\ y)$. Namely, **(4022, 4022), (4046132, 2012), (2012, 4046132), (–4042110, 2010), and (2010, –4042110).**



4. Let a, b, and c be the numbers on the first, second, and third cans knocked down, respectively. Then the score is a + 2b + 3c = 50. From this, it can be seen that a and c are either both even or both odd. There are only 3 possible choices for the value of a, since one of the top row of cans must be knocked down on the first throw.

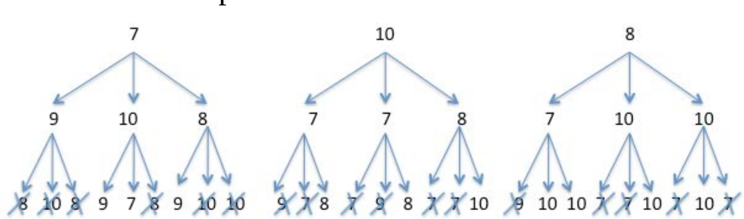
<u>Case 1</u>: a = 7. Since c must also be odd, we cannot have b = 9. There remains two choices for b, namely b = 10 and b = 8. If b = 10, then the score for those two throws is 7 + 2(10) = 27, which leaves 23 for the remaining throw. But the score on the third throw is a multiple of 3, so this case is ruled out. If b = 8, then the score for the two throws is 2 + 2(8) = 23, which leaves a remainder of 27 = 3(9). Since a can numbered 9 is available (and there is only one such can available), case 1 leads to exactly one possible way to score 50.

<u>Case 2</u>: a = 10. Since c must also be even, we must have c = 8 or c = 10.
Case 2a: c = 8. This can only occur if b = 7, giving a score of 10 + 2(7) + 3(8) = 48.
Case 2b: c = 10. This can only occur if b = 8, giving a score of 10 + 2(8) + 3(10) = 56.

<u>Case 3</u>: a = 8. Since c must also be even, we must have c = 10. This can only happen if b = 7 or b = 10. If b = 7, the score is 8 + 2(7) + 3(10) = 52. If b = 10, the score is 8 + 2(10) + 3(10) = 58.

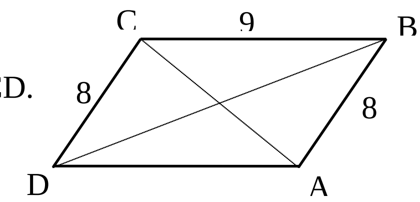
Therefore there is only **one** possible combination that will result in a score of **50**.

Here is a visual representation of the solution.



The numbers crossed in the end are the cases where a and c are not both even or both odd.

5. <u>Method 1</u>:



Use the Law of Cosines on two triangles, ΔABC and ΔBCD.

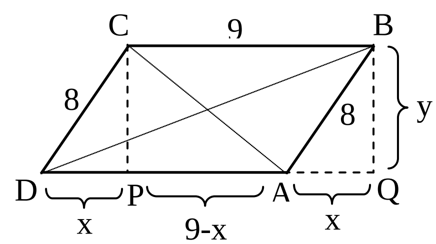
ΔABC: $(AC)^2 = 8^2 + 9^2 - 2(8)(9)\cos\angle ABC$
$(AC)^2 = 145 - 144\cos\angle ABC$

ΔBCD: $(BD)^2 = 8^2 + 9^2 - 2(8)(9)\cos\angle BCD$,
and since $m\angle ABC + m\angle BCD = 180$,
$(BD)^2 = 145 - 144\cos\angle BCD = 145 - 144(-\cos\angle ABC) = 145 + 144\cos\angle ABC$

Adding the two equations $\left.\begin{array}{l}(AC)^2 = 145 - 144\cos\angle ABC \\ (BD)^2 = 145 + 144\cos\angle ABC\end{array}\right\}$ $(AC)^2 + (BD)^2 = 290$

Since we are given that the length of each diagonal is an integer, a quick check tells us that 11 and 13 have squares that sum to 290 and 1 and 17 have squares that sum to 290. However, no triangle with sides of lengths 8, 9, and 17 (or 8, 9, and 1) exists. Therefore, the diagonals have lengths of 11 and 13 and the desired ordered pair is **(11, 13)**.

<u>Method 2</u>:



Construct altitudes as shown.

(i) Using right triangle CPA, $y^2 + (9 - x)^2 = d_1{}^2$

(ii) Using right triangle BQD, $y^2 + (9 + x)^2 = d_2{}^2$

(iii) Using right triangle ABQ, $x^2 + y^2 = 8^2 \Rightarrow y^2 = 8^2 - x^2$

Adding (i) and (ii) and simplifying, we obtain

$$d_1{}^2 + d_2{}^2 = 2x^2 + 162 + 2y^2$$

Substituting (iii) in this last equation and simplifying, we obtain $d_1{}^2 + d_2{}^2 = 290$.

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