THE 2012-2013 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

In addition to scoring student responses based on whether a solution is correct and complete, consideration will be given to elegance, simplicity, originality, and clarity of presentation.

Calculators are <u>NOT</u> permitted.

1. The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a.

   (a) Prove that all of these parabolas pass through a common point and determine the coordinates of this point.

   (b) The vertices of all the parabolas lie on a curve. Find, with proof, the equation of this curve.

2. A sequence of functions is defined by the following rules

   $$\text{(i)}\quad f_1(x) = \frac{2x-1}{x+1} \quad \text{and} \quad \text{(ii)}\quad f_{n+1}(x) = f_1(f_n(x))$$

   for $n = 1, 2, 3, \ldots$. Compute, with proof, $f_{2012}(2013)$.

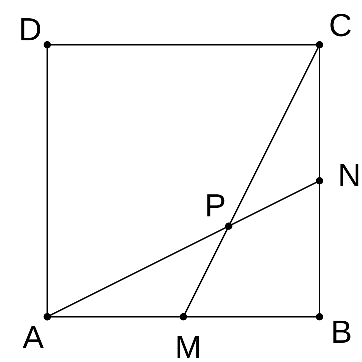
3. Consider the three numbers $6n^2 + 5$, $2n^2 + 3$, and $n^2 + 1$, where n is a positive integer.

   (a) Find, with proof, all values of n for which all three numbers are prime numbers.

   (b) Prove that there are infinitely many values of n for which none of the three numbers is a prime number.

4. Six hotel guests wanted to check out, but the front desk clerk was nowhere to be found. The guests each put their room key on the counter and left. When the clerk returned, he did not know which of the 6 keys went to which room. If the clerk randomly gave the keys to each of the next six guests, compute, with proof, the probability that <u>none</u> of the new guests received the correct room key.

5. In the figure, ABCD is a square, M is the midpoint of $\overline{AB}$, and N is the midpoint of $\overline{BC}$. $\overline{AN}$ and $\overline{CM}$ intersect at point P. Compute, with proof, the ratio of the area of quadrilateral APCD to the area of square ABCD.



THE 2012-2013 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

SOLUTIONS

1. a) We want to show that there is a point (x_1, y_1) which passes through all parabolas with equations of the form $y = x^2 + 2ax + a$. Let m and n be two distinct real numbers. Then we want

$$y_1 = x_1^{\,2} + 2mx_1 + m \quad \text{and} \quad y_1 = x_1^{\,2} + 2nx_1 + n.$$

Subtracting the second equation from the first gives

$$2mx_1 - 2nx_1 + (m-n) = 0 \quad \Rightarrow \quad 2x_1(m-n) = -(m-n).$$

Since $m \neq n$, we must have $x_1 = -\frac{1}{2}$, so $y_1 = \frac{1}{4}$. Thus the desired point is $(-\frac{1}{2}, \frac{1}{4})$.

b) To find the vertex of each parabola $y = x^2 + 2ax + a$, we complete the square, giving $y = x^2 + 2ax + a^2 + a - a^2 = (x+a)^2 + (a - a^2)$. Therefore, the vertex has coordinates $(-a, a - a^2)$. Thus, the vertices lie on a parabola with equation $y = f(x) = -x^2 - x$.

2. Let's see if there is a pattern to the sequence.

$$f_1(x) = \frac{2x-1}{x+1}, \quad f_2(x) = f_1(f_1(x)) = \frac{2\left(\frac{2x-1}{x+1}\right) - 1}{\frac{2x-1}{x+1} + 1} = \frac{x-1}{x}.$$ In a similar manner we find,

$$f_3(x) = f_1(f_2(x)) = \frac{2\left(\frac{x-1}{x}\right) - 1}{\frac{x-1}{x} + 1} = \frac{x-2}{2x-1}.$$ Similarly, $f_4(x) = \frac{1}{1-x}$, $f_5(x) = \frac{x+1}{2-x}$,

$f_6(x) = x$, and $f_7(x) = \frac{2x-1}{x+1}$. So we see that $f_{6n}(x) = x$ for n = 1, 2, 3, ….

Since 2012 = (6)(335) + 2, we have $f_{2012}(2013) = f_2(2013) = \frac{2013-1}{2013} = \frac{2012}{2013}$.

3. By inspection, If $n = 1$ the three numbers are 11, 5, and 2, all prime. If $n = 2$, the three numbers are 29, 11, and 5, again all prime. However when n=3, we get {59, 21, 10}, when $n = 4$ we get {101, 35, 17}, and when $n = 5$ we get {155, 53, 26}, so that each set consists of at least one non-prime. More specifically, each of the sets of three numbers listed above contains a multiple of 5.

(a) We will prove that one of the three numbers must be a multiple of 5 for $n > 2$.

Any integer must have one of the forms $5k$, $5k \pm 1$, or $5k \pm 2$.
If $n = 5k$, $6n^2 + 5 = 150k^2 + 5 = 5(30k^2 + 1)$ which is clearly a multiple of 5.
If $n = 5k \pm 1$, then $2n^2 + 3 = 2(5k \pm 1)^2 + 3 = 50k^2 \pm 20k + 5 = 5(10k^2 \pm 4k + 1)$, again a multiple of 5.
If $n = 5k \pm 2$, then $n^2 + 1 = (5k \pm 2)^2 + 1 = 25k^2 \pm 20k + 5 = 5(4k^2 \pm 4k + 1)$, again a multiple of 5.

In all three cases, therefore, one of the three numbers $6n^2 + 5$, $2n^2 + 3$, and $n^2 + 1$ is a multiple of 5. Since each number is larger than 5 for $n > 2$, the only values of n for which all three numbers are prime are $n = 1$ and $n = 2$.

(b) If n is an odd multiple of 5, $6n^2 + 5$ is a multiple of 5, and $n^2 + 1$ is even.
If n is an odd multiple of 3, $2n^2 + 3$ is a multiple of 3, and $n^2 + 1$ is even.

Hence, if n is any odd multiple of 15, $6n^2 + 5$ is a multiple of 5 (and therefore not prime), $2n^2 + 3$ is a multiple of 3 (and therefore not prime), and $n^2 + 1$ is even (and therefore not prime). Therefore, there are infinitely many values of n for which none of the three numbers is prime.

4. It is easier to compute the probability of the opposite event, namely the one where at least one of the new guests receives the correct room key. Denote that probability by p(B). If the desired probability is p(A), than we compute it as p(A)=1 – p(B).

Let A_k = probability that guest k received the correct room key, A_{kl} = probability that guests k and l got the correct room keys and so on analogously. For example A_{235} denotes the probability that guests 2,3 and 5 received the correct room key. Using the principle of inclusion-exclusion we have

$$p(B) = \sum_{k=1}^{6} A_k - \sum_{\substack{k=1,\\ l>k}}^{6} A_{kl} + \sum_{\substack{k=1,\\ l>k,\\ m>l}}^{6} A_{klm} - \sum A_{klmn} + \sum A_{klmnp} - \sum A_{klmnpq}$$

Notice that $A_1 = A_2 = A_3 = A_4 = A_5 = A_6$ and that $A_{kl} = A_{mn}$ for all choices of k,l,m,n, and so on. Hence,

$$p(B) = 6\,A_1 - \binom{6}{2}A_{12} - \binom{6}{3}A_{123} - \binom{6}{4}A_{1234} - \binom{6}{5}A_{12345} - \binom{6}{6}A_{123456}$$

Now we compute

$$A_1 = \tfrac{5!}{6!}, A_{12} = \tfrac{4!}{6!}, A_{123} = \tfrac{3!}{6!}, A_{1234} = \tfrac{2!}{6!}, A_{12345} = \tfrac{1!}{6!}, A_{123456} = \tfrac{0!}{6!}.$$

Therefore,

$$p(B) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$$

It follows

$$p(A) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{265}{720} = \frac{53}{144} \approx 0.368$$

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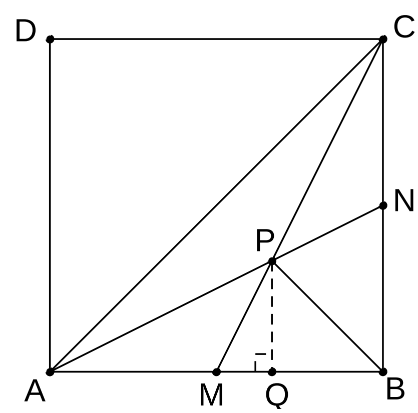
5. Method 1



Construct diagonal $\overline{AC}$. Since $\overline{CM}$ and $\overline{AN}$ are medians of triangle ABC, point P is the centroid of triangle ABC, so that PM = ⅓ CM.

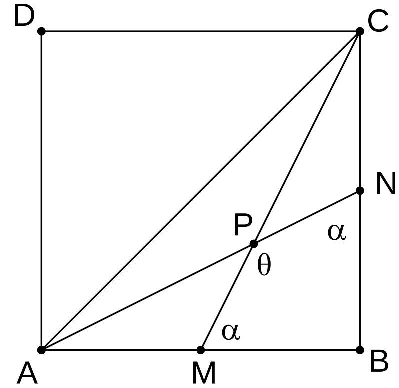
Construct a perpendicular from P to $\overline{AB}$.
ΔMBC and ΔMQP are similar, PQ = ⅓ CB.

Since $\overline{PQ}$ and $\overline{CB}$ are the altitudes of ΔABP and ΔABC, respectively, and they have the same base ($\overline{AB}$), the area of ΔABP is ⅓ the area of ΔABC

Thus, area of ΔABP = ⅓(½ AB^2) = $^1/_6$ (AB^2).

Similarly, the area of ΔCBP = $^1/_6$ (AB^2).
Therefore, Area of quad APCD = Area of ABCD – (area of ABP – area of ΔCBP) = AB^2 – [$^1/_6$ (AB^2) + $^1/_6$ (AB^2)] = ⅔ (AB^2). Therefore, the desired ratio is $\frac{2}{3}$.

Method 2



Construct diagonal $\overline{AC}$. Let m∠CMB = m∠ANB = α and m∠MPN = m∠APC = θ. Using quadrilateral BMPN, θ = 270 – 2α.

Sinθ = sin(270 – 2α) = sin270cos2α – cos270sin2α = –cos2α

Without loss of generality, let CB = 2, so that MB = 1 and CM = $\sqrt{5}$.

Using right ΔMBC, sinα = $\frac{2}{\sqrt{5}}$. Therefore, using the appropriate double angle formula, cos2α = 1- 2sin²α = $1-2\left(\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5}$.

Therefore sinθ = $\frac{3}{5}$

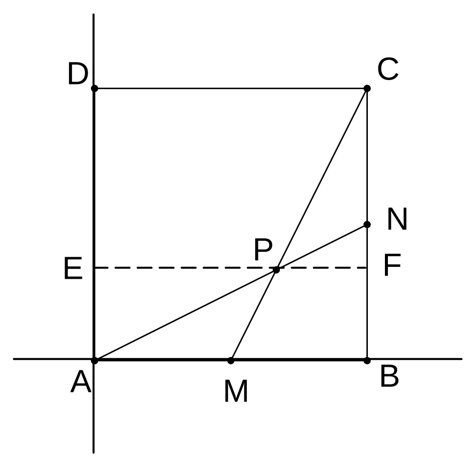
Since $\overline{CM}$ and $\overline{AN}$ are medians of triangle ABC, point P is the centroid of the triangle, so that AP = CP = ⅔ CM = ⅔$\sqrt{5}$.

Therefore, the area of ΔAPC = ½ (AP)(PC)sinθ = ½ (⅔$\sqrt{5}$)(⅔$\sqrt{5}$)($\frac{3}{5}$) = $\frac{2}{3}$.

The area of ΔADC = ½ $(2)^2$ = 2, and the area of quad APCD = 2 + $\frac{2}{3} = \frac{8}{3}$.

Therefore, the desired ratio is $\frac{\frac{8}{3}}{4} = \frac{2}{3}$.

Method 3



Place the figure in the coordinate plane so the vertices of the square have the following coordinates:
A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1).

Then, segment AN falls on the line $y=\frac{1}{2}x$
and segment CM falls on the line $y=2x-1$

Since $\frac{1}{2}x=2x-1 \Rightarrow x=\frac{2}{3}$, the coordinates of P are (2/3, 1/3)

Draw a horizontal line through P that intersects AD at E = (0, 1/3) and BC at F = (1, 1/3)

Since triangle AEP is congruent to triangle CFP, the area of quadrilateral APCD equals the area of rectangle CDEF.

CDEF has an area of 2/3 and the area of ABCD is 1. So, 2/3 is the desired ratio.