THE 2014-2015 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Calculators are <u>NOT</u> permitted

1. Let us say that a person is "acquainted" with another person if <u>each</u> of them knows the other. Suppose there is a gathering of 100 people and that among every four people at the gathering there is at least one who is acquainted with the other three. Prove that in this gathering, there is at least one person who is acquainted with all the remaining 99 people.

2. Suppose that a and b are integers such that $a + 2b$ and $b + 2a$ are squares. Prove that a and b are each multiples of 3.

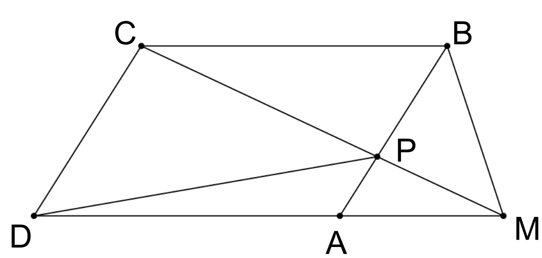
3. Let $P(x)$ be a polynomial with integer coefficients.

   (i) Prove that it is <u>not</u> possible for $P(7) = 11$ and $P(11) = 13$.

   (ii) Suppose that $P(0) = P(1) = 1$. Prove that the equation $P(x) = 0$ has <u>no</u> integer solutions.

4. In the rectangular coordinate system, the line ℓ with equation $2x + y = 4$ is reflected over the y-axis, and its reflection image is named ℓ'. The line m with equation $x + 4y = 8$ is reflected over the x-axis, and its reflection image is named m'. Let the four points of intersection of ℓ and ℓ' with m and m' be A, B, C, and D. Prove that points A, B, C, and D all lie on a circle.

5. Point P is chosen randomly on side AB of parallelogram ABCD. Line segment CP is extended to meet side DA extended at point M, and segment BM is constructed. Prove that the area of triangle BPM is equal to the area of triangle DAP.



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1. Pick any person, say B. If B is acquainted with all the others, we are done.

   Otherwise, there is a person C such that B is not acquainted with C. Then C is not acquainted with B. Randomly choose two other persons, A and D. According to the hypothesis, in the group {A, B, C, D} of four people, either A or D is acquainted with the other three. Without loss of generality, suppose A is acquainted with the other three.

   Now suppose D′ is <u>any</u> other person of the remaining 96. Then either A or D′ is acquainted with the other three in {A, B, C, D′}. Therefore, A and D′ are acquainted (remember the definition of acquainted says that it is symmetric). Thus A is acquainted with all the remaining 99 people.

2. Let $a + 2b = m^2$ and $2a + b = n^2$, where m and n are integers. Then $m^2 + n^2 = 3a + 3b$, and so $m^2 + n^2$ is a multiple of 3. Any integer m can be written in the form $3k$, $3k + 1$, or $3k – 1$, where k is some integer. If $m = 3k$, then m^2 is a multiple of 9. If $m = 3k + 1$ or $3k – 1$, then $m^2 = 9k^2 \pm 6k + 1 = 3(3k^2 \pm 2k) + 1$ and thus it is one more than a multiple of 3. Similarly, n^2 is either a multiple of 9 or one more than a multiple of 3.

   If either m^2 or n^2 is not a multiple of 9, then it follows that $m^2 + n^2$ is either one or two more than a multiple of 3. However, we have already shown that $m^2 + n^2$ is a multiple of 3. Therefore, both m^2 and n^2 are multiples of 9.

   Since $m^2 + n^2 = 3(a + b)$ is a multiple of 9, $a + b$ is a multiple of 3. Also, $a – b = n^2 – m^2$ is a multiple of 3. It follows that $2a = (a + b) + (a – b)$ is a multiple of 3, and hence a is a multiple of 3. Also $b = (a + b) – a$ is a multiple of 3. Hence both a and b are multiples of 3.

3. (i) Let $P(x) = \sum_{k=0}^{n} a_k x^k$. If P(7) = 11 and P(11) = 13, then

$$P(11) – P(7) = \sum_{k=1}^{n} a_k (11^k - 7^k) = 13 – 11 = 2.$$

   Since $x – y$ is a factor of $x^k - y^k$, then $\sum_{k=1}^{n} a_k (11^k - 7^k)$ is divisible by 11 – 7 = 4.

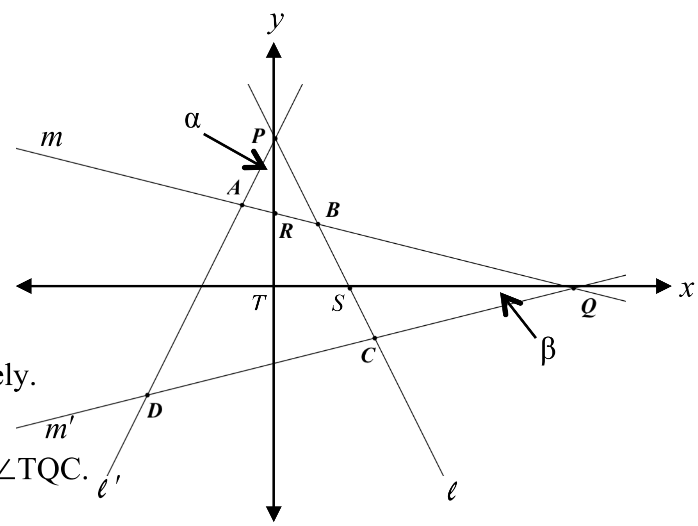
   Therefore, 2 must be divisible by 4, which is not possible.

   (ii) Suppose b is an integer root. Then $P(b) = 0$ and $P(x) = (x – b)f(x)$, where $f(x)$ is a polynomial with integer coefficients. Thus, $1 – b$ divides P(1) = 1. Therefore, either $1 – b = 1$ or $1 – b = –1$.

   If $1 – b = 1$, then $b = 0$, which implies P(0) = 0. Contradiction, since P(0) = 1.

   If $1 – b = –1$, then $b = 2$ and $P(b)$ is a sum of powers of 2 and a constant term c. Since $P(b) = 0$, c must be even. But $P(0) = c$, so P(0) must be even. Contradiction, since P(0) = 1.

4. Method 1 (elegant and simple!!)



We must show that quadrilateral ABCD is cyclic.

The lines intersect on the x and y-axes as shown, and the axes bisect the angles formed. Represent the origin as point T, and the intersections of ℓ and m with the x and y-axes as S and R, respectively.

Let $\alpha = m\angle APR = m\angle RPB$ and $\beta = m\angle BQT = m\angle TQC$.

Since $\angle DAR$ is an exterior angle of ΔAPR, we have $m\angle DAR = \alpha + m\angle ARP$.
Also, $m\angle ARP = m\angle TRQ$. Since ΔTRQ is a right triangle,
$m\angle TRQ = 90 - m\angle RQT \Rightarrow m\angle ARP = 90 - \beta$. Therefore, $m\angle DAR = \alpha + 90 - \beta$.

Similarly, using ΔSCQ, we can show that $m\angle DCB = \beta + 90 - \alpha$.

Thus, $m\angle DAR + m\angle DCB = (\alpha + 90 - \beta) + (\beta + 90 - \alpha) = 180$. Since these are a pair of opposite angles of quadrilateral ABCD, and they are supplementary, the quadrilateral is cyclic.

Method 2 (brute force!!)

The equation of line ℓ' is $-2x + y = 4$ and the equation of line m' is $x - 4y = 8$.
By solving the four pairs of equations simultaneously, we obtain the coordinates of the four points, as pictured above: $A\left(\frac{-8}{9}, \frac{20}{9}\right)$, $B\left(\frac{8}{7}, \frac{12}{7}\right)$, $C\left(\frac{24}{9}, \frac{-12}{9}\right)$, and $D\left(\frac{-24}{7}, \frac{-20}{7}\right)$.
If the points were to lie on a circle, then quadrilateral ABCD would be cyclic. The sides of the quadrilateral would be chords of the circle, and so would the diagonals. The center of the circle would be the intersection of the perpendicular bisectors of any two of these chords.
We will use diagonals $\overline{AC}$ and $\overline{BD}$, whose slopes are -1 and 1, respectively.
The midpoint of $\overline{AC}$ has coordinates $\left(\frac{8}{9}, \frac{4}{9}\right)$. Therefore, the equation of its perpendicular bisector is $y - \frac{4}{9} = 1\left(x - \frac{8}{9}\right)$ or $9x - 9y = 4$.
The midpoint of $\overline{BD}$ has coordinates $\left(\frac{-8}{7}, \frac{-4}{7}\right)$. Therefore, the equation of its perpendicular bisector is $y + \frac{4}{7} = -1\left(x + \frac{8}{7}\right)$ or $7x + 7y = -12$.

The common solution to these two equations (the hypothetical center of the circle) is $P\left(\frac{-40}{63}, \frac{-68}{63}\right)$.

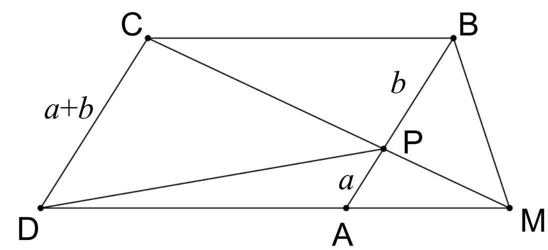
Now, rewriting the coordinates of points A, B, C, and D with a common denominator 63,

$$A\left(\frac{-56}{63}, \frac{140}{63}\right),\quad B\left(\frac{72}{63}, \frac{108}{63}\right),\quad C\left(\frac{168}{63}, \frac{-84}{63}\right), \text{ and } D\left(\frac{-216}{63}, \frac{-180}{63}\right),$$

and using the distance formula to find the lengths of $\overline{AP}$, $\overline{BP}$, $\overline{CP}$, and $\overline{DP}$, we obtain:

$$AP = \sqrt{\frac{16^2}{63^2} + \frac{(-208)^2}{63^2}},\ BP = \sqrt{\frac{112^2}{63^2} + \frac{176^2}{63^2}},\ CP = \sqrt{\frac{208^2}{63^2} + \frac{(-16)^2}{63^2}},\ DP = \sqrt{\frac{176^2}{63^2} + \frac{112^2}{63^2}}.$$

Since $16^2 + 208^2 = 43520 = 112^2 + 176^2$, A, B, C, and D are all the same distance from P. Therefore, all four points lie on a circle.



5. Let AP = a and BP = b, so that CD = $a + b$. Since PA is parallel to CD, ΔCDM is similar to ΔPAM. Therefore,

$$\frac{DM}{AM} = \frac{CD}{PA} \quad \text{or} \quad \frac{DM}{AM} = \frac{a+b}{a}.$$

Since DM = DA + AM, this last equation can be rewritten as

$$\frac{AM + AD}{AM} = \frac{a+b}{a} \quad \Rightarrow \quad 1 + \frac{DA}{AM} = 1 + \frac{b}{a} \quad \Rightarrow \quad \frac{DA}{AM} = \frac{b}{a}.$$

Consider ΔDAP and ΔPAM. They have the same altitude from point P.
Therefore the ratio of their areas is equal to the ratio of their bases (DA and AM).

Thus, $\frac{\text{area of } \Delta DAP}{\text{area of } \Delta PAM} = \frac{DA}{AM} = \frac{b}{a}$

Similarly, ΔBPM and ΔPAM have the same altitude from point M, so the ratio of their areas is equal to the ratio of their bases (BP and PA).

Thus, $\frac{\text{area of } \Delta BPM}{\text{area of } \Delta PAM} = \frac{BP}{AM} = \frac{b}{a}$.

Therefore, $\frac{\text{area of } \Delta DAP}{\text{area of } \Delta PAM} = \frac{\text{area of } \Delta BPM}{\text{area of } \Delta PAM}$ from which the area of triangle DAP and the area of triangle BPM are equal.