THE 2015-2016 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Kennesaw State University
College of Science and Mathematics
Department of Mathematics

Calculators are NOT permitted

Time allowed: 2 hours

1. Let $P = 1 \cdot 2 \cdot 3 \cdots n = n!$ and let $S = 1 + 2 + 3 + \ldots + n$, where n is a positive integer.

   (a) Prove that if n is odd, then S divides P exactly.

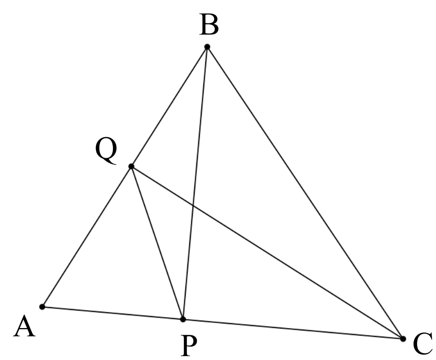
   (b) Prove that the converse of part (a) is not true.

2. Let $x_1, x_2, x_3, \ldots x_7$ be real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 2015 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 2016 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 2017. \end{aligned}$$

   Compute, with proof, the value of $16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$.

3. In acute triangle ABC, altitudes $\overline{BP}$ and $\overline{CQ}$ are constructed, with P on $\overline{AC}$ and Q on $\overline{AB}$. If BC = 8 and PC = 5, compute, with proof, cos ($\angle$PQC).

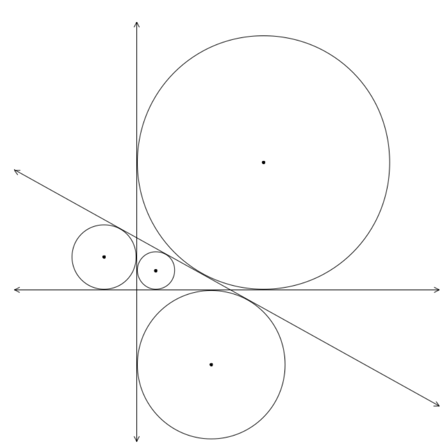


4. Let $f(x) = ax^2 + bx + c$ be a polynomial with real coefficients.

   (a) If $f(-1)$, $f(0)$, and $f(1)$ are all even integers, prove that $f(k)$ is an even integer for every integer k.

   (b) Suppose there exist three consecutive integers m, n, p for which $f(m)$, $f(n)$ and $f(p)$ are all even integers. Prove that $f(k)$ is an even integer for every integer k.

5. The diagram shown is formed by extending the sides of a right triangle and constructing the four circles that are each tangent to all three lines. Prove that the sum of the lengths of the radii of the three smaller circles is equal to the length of the radius of the largest circle.



SOLUTIONS – KSU MATHEMATICS COMPETITION – PART II 2015-16

1. (a) If n is odd, then $n = 2k + 1$ where k is a positive integer. Then

$$\frac{P}{S} = \frac{1\cdot 2\cdot 3\cdots(2k+1)}{1+2+3+...+(2k+1)} = \frac{(2k+1)!}{\frac{1}{2}(2k+1)(2k+2)} = \frac{(2k+1)!}{(2k+1)(k+1)}$$

Since both $(2k + 1)$ and $(k + 1)$ are factors of $(2k + 1)!$, S divides P.

(b) To prove that the converse of part (a) is <u>not</u> true, we need only find a counterexample. That is, we need to find an even integer n such that S divides P exactly.

Using a little trial and error, if $n = 8$, $P = 8! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8$, and $S = 36 = 2\cdot3\cdot6$. Clearly S divides P for $n = 8$, and the converse of part (a) is not true.

2. Observe that subtracting 3 times the second equation from the sum of the first equation and 3 times the third equation gives an equation with the desired expression on the left.

More generally, we can deduce this relation by finding coefficients a, b, and c such that $an^2 + b(n + 1)^2 + c(n + 2)^2 = (n + 3)^2$ holds for all n.

Considering this last equation as a polynomial identity in n, expand and simplify both sides, then equate coefficients of like powers of n.

$$\begin{aligned} a + b + c &= 1 \\ 2b + 4c &= 6 \\ b + 4c &= 9 \end{aligned}$$

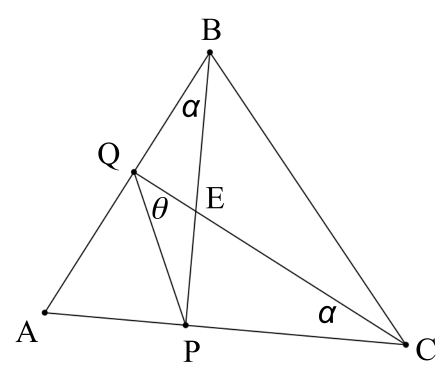
The solution of this system is $a = 1$, $b = -3$, and $c = 3$. Therefore, the desired value is

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7 = 1(2015) - 3(2016) + 3(2017) = 2018.$$

3. We will prove $\angle PQC \cong \angle PBC$.

<u>Method 1</u>: Let E be the point of intersection of the altitudes, and let $m\angle PQC = \theta$. Since ΔBEQ is similar to ΔPEC, $\angle ABP \cong \angle ACQ$. Let $m\angle ABP = m\angle ACQ = \alpha$.

Then $\sin\angle BQP = \sin(90 + \theta) = \cos\theta$.



Using the Law of sines on ΔBQP:

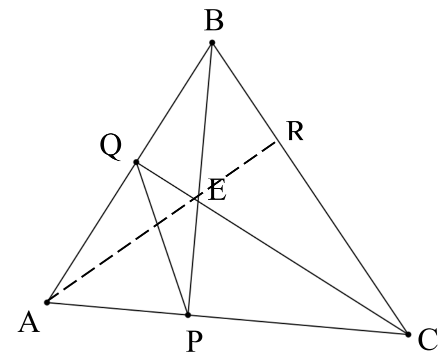
$$\frac{\sin\angle BQP}{BP} = \frac{\sin\alpha}{QP}$$

Using the Law of sines on ΔQCP:

$$\frac{\sin\alpha}{\text{QP}} = \frac{\sin\theta}{\text{PC}}$$

Therefore, $\dfrac{\sin\angle\text{BQP}}{\text{BP}} = \dfrac{\sin\theta}{\text{PC}}$. $\Rightarrow$ $\sin\theta = \dfrac{\text{PC}}{\text{PB}}\sin\angle\text{BQP}$ $\Rightarrow$ $\sin\theta = \dfrac{\text{PC}}{\text{PB}}\cos\theta$

Thus, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{PC}{PB} = \tan\angle PBC \Rightarrow \theta = \angle\text{PBC}$.



Method 2: Let E be the point of intersection of the altitudes. Construct the third altitude to point R on side $\overline{\text{BC}}$. Since the altitudes of a triangle are concurrent, $\overline{\text{AR}}$ passes through point E.

Right angles AQE and APE are supplementary, making quadrilateral AQEP cyclic. Therefore, ∠PQE and ∠EAP are congruent inscribed angles.

Triangle APE and triangle BRE are similar, and thus ∠EAP ≅ ∠EBR. Then by the transitive property, ∠PQE (PQC) ≅ ∠EBR (PBC)

Since BC = 8 and PC= 5, using the Pythagorean Theorem on ΔBPC, BP = $\sqrt{39}$.

Therefore, cos (∠PQC) = cos(∠PBC) = $\dfrac{\sqrt{39}}{8}$.

4. (a) We are given that, for $f(x) = ax^2 + bx + c$, and $f(-1)$, $f(0)$, and $f(1)$ are even integers. Thus

$$\begin{aligned} f(-1) &= a - b + c = 2r \\ f(0) &= \quad\quad\quad\; c = 2s \\ f(1) &= a + b + c = 2t \end{aligned}$$

where r, s, and t are integers. Solving for a, b, and c in terms of r, s, and t, we obtain

$$\begin{aligned} a &= r + t - 2s \\ b &= -r + t \\ c &= \quad\quad\;\; 2s \end{aligned}$$

Therefore, a, b, and c are all integers and c is even. Also, since $a - b = 2r - 2s$, which is an even integer, we know that a and b are either both even or both odd.

Case (i) a and b are both even.
Then for every integer k, $f(k) = ak^2 + bk + c$ is an even integer since ak^2, bk, and c are all even integers.

Case (ii) a and b are both odd.

Then for every even integer k, ak^2 and bk are even, making $f(k) = ak^2 + bk + c$ even.

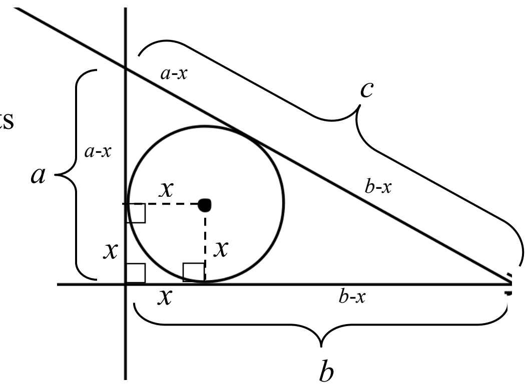
For every <u>odd</u> integer k, $ak^2 + bk$ is the sum of two odd integers, and therefore $ak^2 + bk$ is an even integer. So $f(k) = (ak^2 + bk) + c$ is an even integer.

(b) Let the given set of three consecutive integers be $t - 1$, t, and $t + 1$, so that $f(t - 1)$, $f(t)$, and $f(t + 1)$ are all even integers. Let $g(x) = f(x + t)$. Then

$$g(-1) = f(t - 1)$$
$$g(0) = f(t)$$
$$g(1) = f(t + 1)$$

Since these are all even integers, $g(x)$ is a quadratic polynomial satisfying the conditions of part (a). Therefore, $g(h)$ is even for all integers h. Letting $h = k - t$, we have $g(k - t) = f(k)$ is an even integer for every integer k.

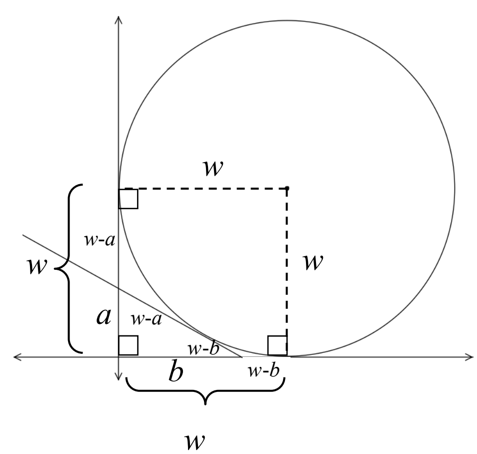
5. First consider the inner circle. Noting that the tangents to a circle from an external point are congruent, label the diagram as indicated. Then if the lengths of the sides of the right triangle are a, b, and c,



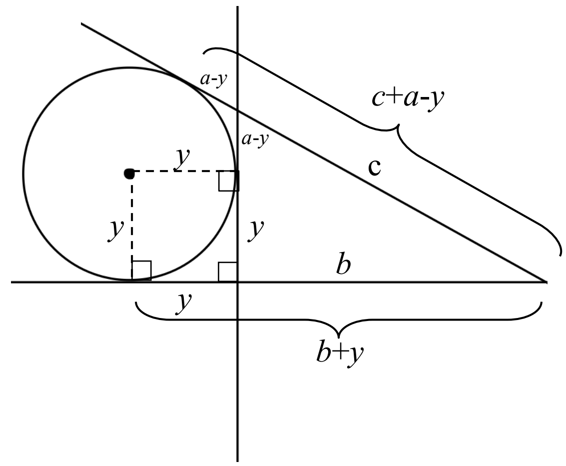
$(a - x) + (b - x) = c$ so that $x = \dfrac{a - c + b}{2}$.

Next consider the largest circle and label the diagram as indicated. Then

$(w - a) + (w - b) = \mathrm{c}$ so that $w = \dfrac{a + b + c}{2}$.



Next consider either of the remaining two circles and label the diagram as indicated. Then



$c+a-y=b+y$ so that $y=\dfrac{c+a-b}{2}$.

Finally, using the same argument on the remaining circle with a and b interchanged gives a radius of length $\dfrac{c+b-a}{2}$.

Adding the lengths of the radii of the three smaller circles, $\dfrac{a-c+b}{2}+\dfrac{c+a-b}{2}+\dfrac{c+b-a}{2}=\dfrac{a+b+c}{2}$ which is the same as the radius of the largest circle.