THE 2017–2018 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART I – MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.

NO CALCULATORS

90 MINUTES

1. Huckleberry High School held a dance for its students. The first N students paid \$10 each for a ticket to the dance. But then the ticket price was reduced. After the price reduction, the number of students buying tickets was 50% more than before the price reduction, but the amount of money received by the school was only 20% more than before the price reduction. What was the amount of the reduced ticket price?

   (A) \$8.00 (B) \$7.50 (C) \$7.20 (D) \$7.00 (E) \$6.40

2. The absolute value of a certain number N is $3\frac{1}{2}$ more than the number itself. How much more than the number N is its reciprocal?

   (A) $\frac{13}{9}$ (B) $\frac{21}{16}$ (C) $\frac{33}{28}$ (D) $\frac{36}{25}$ (E) None of these

3. A palindrome is a word or a number (like RADAR or 1221) which reads the same forwards and backwards. If dates are written in the format MMDDYY (i.e. 2 digits for month, 2 digits for day, and 2 digits for year), how many dates in this century are palindromes?

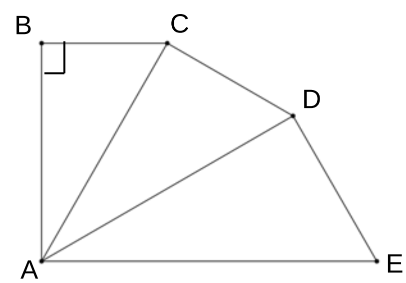
   (A) 12 (B) 16 (C) 24 (D) 36 (E) 144

4. In planning a group activity for his statistics class, Dr. Garner found that the number of possible groups of 4 students in his class is exactly four times the number of possible groups of 3 students. How many students are in Dr. Garner's class?

   (A) 16 (B) 17 (C) 18 (D) 19 (E) 20

5. In the diagram, $\angle B$ is a right angle, $\overline{BC}$ is parallel to $\overline{AE}$, and $\Delta BCA \sim \Delta CDA \sim \Delta DEA$. Compute the ratio $BC : AE$.

   (A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{3}{7}$ (D) $\frac{3}{8}$ (E) $\frac{4}{9}$



6. The integers b and c are chosen so that

(i) one of the roots of the quadratic equation $5x^2 + bx + c = 0$ is 2 and

(ii) one of the roots of the quadratic equation $5x^2 + cx + b = 0$ is 3.

Compute the sum of the other two roots.

(A) $-\frac{7}{5}$ (B) $-\frac{8}{5}$ (C) $\frac{1}{5}$ (D) $\frac{7}{5}$ (E) $\frac{8}{5}$

7. If log(K) + (log4)(log4) = (log40)(log40), compute K.

(A) 40 (B) 64 (C) 80 (D) 128 (E) 160

8. The perimeter of parallelogram ABCD is 40 and its two altitudes are 4 and 7. Which of the following is closest to the value of $\sin \mathrm{A}$?

(A) .45 (B) .52 (C) .55 (D) .57 (E) .65

9. Mrs. Brewster purchased two mugs and three plates. When paying, Mrs. Brewster noticed that the sales clerk multiplied the total price of the mugs by the total price of the plates instead of adding them (all prices in dollars). Amazingly, the sales clerk arrived at the correct cost for the whole purchase, \$4.05. Which of the following could be the total price for one mug and one plate?

(A) \$1.30 (B) \$1.45 (C) \$1.50 (D) \$1.60 (E) \$1.65

10. Let N be a two digit positive integer whose value is increased by 75% when its digits are reversed. Compute the sum of all such two-digit numbers N.

(A) 36 (B) 60 (C) 72 (D) 120 (E) 144

11. Let f be a function defined as follows: $f(1/x) + 2f(x) = x$ for all real $x \neq 0$. Compute $f(5)$.

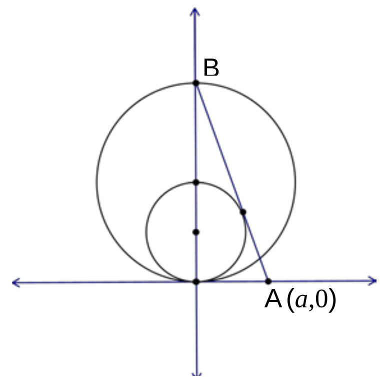
(A) $\frac{1}{5}$ (B) $\frac{49}{15}$ (C) 5 (D) $\frac{49}{5}$ (E) $\frac{5}{49}$

12. Let p, q, and r be the roots of the equation $x^3 - 12x^2 + 9x + c = 0$. If p, q, and r, in that order, form a geometric sequence, compute the value of q.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) None of these

13. For how many positive integers n will $\frac{(n+1)^2}{n+12}$ be an integer.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5



14. In the diagram, two circles of radii 1 and 2 are tangent to each other and the positive *x*-axis at the origin. A segment that is tangent to the smaller circle is drawn from point B, the *y*-intercept of the larger circle, to point A on the positive x-axis. If the coordinates of point A are $(a,0)$, compute the value of a .

(A) 1 (B) $\sqrt{2}$ (C) 1.5 (D) $\sqrt{3}$ (E) 2

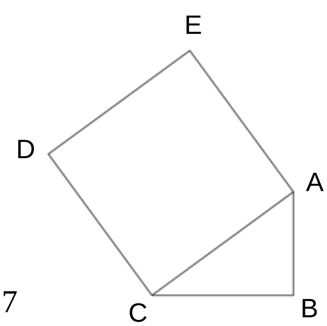
15. Let a ⓖ b represent the operation on two number a and b which selects the greater of the two numbers, with a ⓖ $a = a$. Let a ⓢ b represent the operation which selects the smaller of the two numbers, with a ⓢ $a = a$. If a, b, and c are distinct numbers and a ⓢ (b ⓢ c) = (a ⓢ b) ⓖ (a ⓢ c), which of the following must be true?

(A) $a < b$ and $a < c$ (B) $a > b$ and $a > c$ (C) $c < b < a$

(D) $c < a < b$ (E) $a < b <$ c

16. One rainy afternoon, Debbie got bored and decided to write some numbers in a row. She wrote the number 1 once, then the number 2 twice, then the number 3 three times, and so on until she had written the number 99 ninety-nine times. What was the 2017th digit that Debbie wrote?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5



17. In the diagram, ABC is a right triangle with right angle at B. ACDE is a square constructed on AC. If AB = 7 and BC = 8, compute the distance from B to D.

(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

18. The integers from 1 to 100 are written on 100 separate slips of paper and placed in a box. Three of these slips are drawn at random, without replacement. Compute the probability that the numbers on the slips that are drawn can be arranged to form an arithmetic sequence.

(A) $\frac{3}{100}$ (B) $\frac{1}{66}$ (C) $\frac{1}{396}$ (D) $\frac{17}{3300}$ (E) None of these

19. Consider the following infinite series

$$S = 1 - 2 - 3 + 4 + 5 - 6 - 7 + 8 + 9 - 10 - 11 + 12 + 13 - 14 - 15 + \ldots$$

Define the partial sums of the series as follows:

$$S_1 = 1,\ S_2 = 1 - 2,\ S_3 = 1 - 2 - 3,\ S_4 = 1 - 2 - 3 + 4,\ S_5 = 1 - 2 - 3 + 4 + 5,$$

and so on. For what value of n does S_n exceed 2017 for the first time?

(A) 2017 (B) 2018 (C)) 2019 (D) 2020 (E) 2021

20. Each of the following numbers is a prime, and all but one of them can be expressed as the difference of the cubes of two positive integers. Which one cannot?

(A) 25,366,109 (B) 2,465,227 (C) 58,887,991 (D) 25,675,651 (E) 1,114,471

21. For what value of n will a regular n-gon inscribed in a circle of radius 3 have an area of 3^3?

(A) 6 (B) 8 (C) 9 (D) 12 (E) 15

22. At one point in a class vote for student council representative, Don learned that exactly 45% of those voting had voted for him. After another five minutes of voting he had only 30% of the vote. The least number of people that could have voted during the five minute period is:

(A) 7 (B) 9 (C) 10 (D) 20 (E) 25

23. In a certain isosceles trapezoid, the length of the longer base is equal to the length of a diagonal, and the length of the shorter base is equal to the length of the altitude. Compute the ratio of the length of the shorter base to that of the longer base.

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{2}{3}$ (D) $\frac{4}{7}$ (E) $\frac{5}{8}$

24. If x, y, and z are positive integers, with $2x + 2y + z = 2017$ and $x + 2y + 2z = 2018$, what is the smallest possible value of $x + y + z$?

(A) 1006 (B) 1007 (C) 1008 (D) 1009 (E) 1010

25. In triangle ABC, the length of side $\overline{AB}$ is 7, and the other two side lengths are integers. Point D lies on $\overline{AC}$ between A and C, such that AD = 3 and BD = 5. Which of the following is a possible length for $\overline{BC}$?

(A) 13 (B) 15 (C) 17 (D) 19 (E) None of these

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Solutions

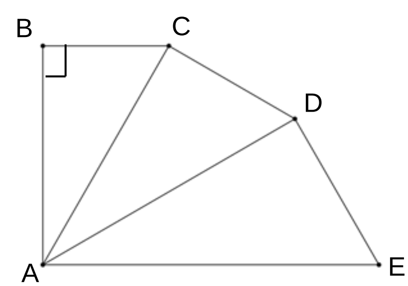
1. A Let x = the reduced ticket price. Then 1.2(10N) = 1.5Nx. Solving, $x = \frac{12}{1.5} = 8$.

2. C Let N = the number. N is negative, for if it wasn't, its absolute value would not be more than N. Therefore, $-N = N + 3.5$ and $N = -1.75 = -\frac{7}{4}$. Thus, taking the reciprocal increases N by $-\frac{4}{7} - \left(-\frac{7}{4}\right) = \frac{33}{28}$.

3. C Only days 11 and 22 can be the middle two digits (since no month has 33 days). Since there are 12 months in a year, the first two digits can only be 01, 02, 03, …, 12. Thus there are 24 possibilities.

4. D The number of groups of four students is ${}_nC_4 = \frac{n!}{4!\,(n-4)!}$. The number of groups of three students is ${}_nC_3 = \frac{n!}{3!\,(n-3)!}$. Therefore, $\frac{n!}{4!\,(n-4)!} = \frac{4n!}{3!\,(n-3)!}$. Simplifying and solving for n gives $n = 19$.

5. D Because $\overline{BC}$ is parallel to $\overline{AE}$, $\angle BAE$ is also a right angle. Since the corresponding angles of similar triangles are congruent, $m\angle BAC = m\angle CAD = m\angle DAE = 30°$, making each triangle a 30-60-90 triangle. Letting BC = a, AC = $2a$, AD = $\frac{2(AC)}{\sqrt{3}} = \frac{4a}{\sqrt{3}}$, and AE = $\frac{2(AD)}{\sqrt{3}} = \frac{8a}{3}$. Thus, the desired ratio BC : AE = $\frac{a}{\frac{8a}{3}} = \frac{3}{8}$.



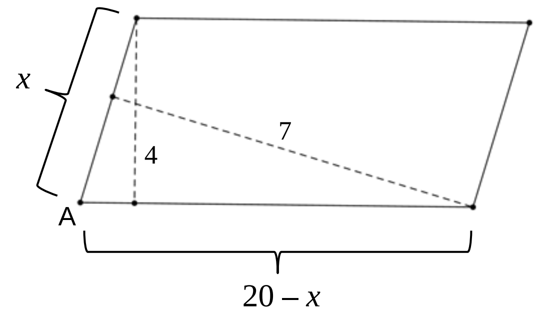
6. B Substituting, we obtain two equations: $20 + 2b + c = 0$ and $45 + 3c + b = 0$. Solving these two equations, we obtain $b = -3$ and $c = -14$. Therefore, the first equation is $5x^2 - 3x - 14 = 0$, whose roots are 2 and $-\frac{7}{5}$. The second equation is $5x^2 - 14x - 3 = 0$, whose roots are 3 and $-\frac{1}{5}$. The desired sum is $-\frac{8}{5}$.

7. E The equation becomes $\log(K) = (\log 40)^2 - (\log 4)^2$. Factoring the right side, $\log(K) = [\log(40) - (\log 4)][\log(40) + (\log 4)]$. Using the properties of logarithms, $\log(K) = \log\left(\frac{40}{4}\right)\log(40 \cdot 4) = \log(160)$. Therefore $K = 160$.

8. C Since consecutive angles of a parallelogram are supplementary, each angle of the parallelogram has the same sine.
Let K = the area of the parallelogram. Then

$K = 7x = 4(20 - x) \quad \Rightarrow \quad x = \dfrac{80}{11}$.

Thus $\sin A = \dfrac{4}{x} = \dfrac{11}{20} = .55$.



9. E Let m be the cost of one mug and p be the cost of one plate. We know that $2m + 3p = 4.05$ and $(2m)(3p) = 4.05$. From the first equation, $2m = 4.05 - 3p$.

Method 1: Substituting into the second equation,
$(4.05 - 3p)(3p) = 4.05 \Rightarrow 9p^2 - 12.15p + 4.05 = 0 \Rightarrow 900p^2 - 1215p + 405 = 0$
Dividing by 45, $20p^2 - 27p + 9 = 0 \Rightarrow (4p - 3)(5p - 3)$, so that $p = \dfrac{3}{4}, \dfrac{3}{5}$. If $p = \dfrac{3}{5}$, then m is not an integer. If $p = \dfrac{3}{4} = .75$, then $m = .90$. Therefore, Therefore, the desired total is \$1.65.

Method 2: Since $2m + 3p = 4.05$ and $(2m)(3p) = 4.05$, we have $2m + 3p = 6mp$ and $p = \dfrac{2m}{3(2m-1)}$. If $m = \$1$, then $p = \$\left(\dfrac{2}{3}\right)$, and the total is \$4, not enough. If $m = \$.75$, then $p = \$1$, and the total is \$4.50, too much. Thus, we can conclude that $.75 < m < 1.00$. Trying a few values, we find that $m = .90$ and $p = .75$ works. Therefore, the desired total is \$1.65.

10. D Represent such a number N as $10t + u$. Then

$$10u + t = 10t + u + \frac{3}{4}(10t + u)$$
$$4(10u + t) = 4(10t + u) + 30t + 3u$$
$$40u + 4t = 70t + 7u \quad \Rightarrow \quad 33u = 66t \quad \Rightarrow \quad u = 2t$$

Therefore, the possible values of N are 12, 24, 36, and 48, and the desired sum is 120.

11. B We are given $f(1/x) + 2f(x) = x$ for all real $x \neq 0$. Substituting $1/x$ for x, we obtain $f(x) + 2f(1/x) = 1/x$. Multiplying the original equation by 2 and subtracting the two equations, $3f(x) = 2x - (1/x) = \dfrac{2x^2 - 1}{x}$. Thus, $f(x) = \dfrac{2x^2 - 1}{3x}$ and $f(5) = \dfrac{49}{15}$.

12. C In any equation of the form $x^3 + ax^2 + bx + c = 0$ with roots p, q, and r,
(i) $p + q + r = -a$, (ii) $pq + pr + qr = b$, and (iii) $pqr = -c$

Represent the three roots of the given equation $x^3 - 12x^2 + 9x + c = 0$ by $p = a$, $q = at$, and $q = at^2$.

Using (ii) $a(at) + a(at^2) + (at)(at^2) = a^2t(1 + t + t^2) = 9$.

Using (i) $a + at + at^2 = a(1 + t + t^2) = 12$

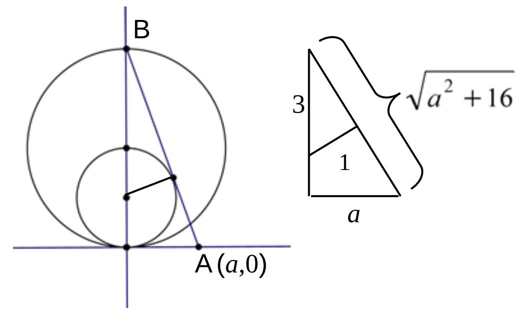
Therefore, $\dfrac{9}{a^2t} = \dfrac{12}{a} \Rightarrow 9 = 12at \Rightarrow at = q = \dfrac{3}{4}$.

13. A Performing the division $\dfrac{n^2 + 2n + 1}{n+12}$ we obtain $n - 10 + \dfrac{121}{n+12}$. We need only find values of n for which $\dfrac{121}{n+12}$ is an integer. Thus, $n + 12 = \pm1, \pm11, \pm121$. Only one of these, $n + 12 = 121$ yields a positive integer for n.

14. B The center of the smaller circle has coordinates $(0, \frac{1}{2})$.

The two right triangles shown are similar.

Therefore, $\dfrac{1}{a} = \dfrac{3}{\sqrt{a^2+16}}$. Solving gives $a = \sqrt{2}$.



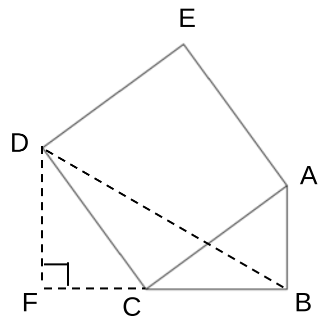
15. A We are given $a\,s\,(b\,s\,c) = (a\,s\,b)\,g\,(a\,s\,c)$, where a, b, and c are distinct. Clearly, the left side of this equation is always the smallest of the three numbers a, b, and c. If the smallest of the three numbers is b, then the left side is b, while the right side simplifies to $b\,g\,(a\,s\,c)$, which is definitely a or c. Thus b cannot be the smallest of the three numbers. Similarly, c cannot be the smallest of the three numbers. This means that a is smallest. This eliminates all but choices (A) or (E) from the set of possible answers. Because of the symmetry in the equation regarding b and c, neither one need be less than the other, so choice (E) may or may not be true. However, the problem asks which of the choices must be true, so only choice (A) is acceptable.

16. E It is easier to keep count if instead of writing 1223334444…, we insert zeros before all the one-digit numbers to get 0102020303030404040404…, so that all the numbers written can be thought of as two-digit numbers. This adds one extra digit to each one-digit number, for a total of 1 + 2 + 3 + … + 9 = 45 extra digits. So instead of looking for the 2017th digit, we are looking for the 2017 + 45 = 2062nd digit. Since each number written is two digits long, the 2062nd digit is the second digit of the 1031st two-digit number.

Thus we need the smallest value of n for which $\dfrac{n(n+1)}{2} \geq 1031$ or $n^2 + n \geq 2062$.

Since $44^2 + 44 = 1980$ and $45^2 + 45 = 2070$, the 2017th digit is the 2nd digit of 45, or 5.

17. E Extend BC through C and construct a perpendicular from D to BC extended at F. Then right triangle ABC is congruent to right triangle CFD (AC ≅ CD, <ACB ≅ <CDF). Since DF = BC = 8, and CF = AB = 7, we can use the Pythagorean Theorem on triangle DFB. Therefore,
BD = $\sqrt{8^2+15^2}=\sqrt{289}=17$.



18. B There are 98 AP's whose common difference is one, i.e. 1,2,3; 2,3,4; …; 98,99,100.
There are 96 AP's whose common difference is 2, i.e. 1,3,5; 2;4;6; …;96,98,100.
There are 94 AP's whose common difference is 3, i.e. 1,4,7;,2,5,8; …; 94,97,100.
……………………………………………………………………………………
Finally, there are two AP's whose common difference is 49, i.e. 1,50,99; 2,51,100.
Each of these AP's can be drawn in six different ways, that is 1,2,3 can be drawn as 1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2; 3,2,1.
Therefore, there are 6(2+4+6+ … +98) = 6(50)(49) ways to draw the arithmetic sequences, and there are (100)(99)(98) possible ways of drawing any of the three numbers. The desired probability is $\dfrac{(6)(50)(49)}{(100)(99)(98)}=\dfrac{1}{66}$.

19. E Calculating the values of the first few partial sums, we obtain

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
1	-1	-4	0	5	-1	-8	0	9	-1	-12	0	13	-1	-16

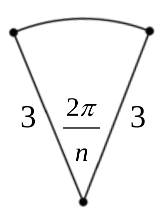
From this we see that S_n = 0 when n is a multiple of 4, and $S_n = n$ when n is one more than a multiple of 4. Thus, S_{2016} = 0, S_{2017} = 2017, S_{2018} and S_{2019} are negative, S_{2020} = 0, and S_{2021} =2021 is the first to exceed 2017. Therefore, n = 2021.

20. A Consider the equation $x^3-y^3=p$, where x and y are positive integers and p is a prime. Since the left side of this equation factors into $(x-y)(x^2+xy+y^2)$, but the right side is prime, the only possibility is $x-y=1$. Substituting $x=y+1$, we get

$$p=(y+1)^3-y^3=3y^2+3y+1$$

Thus, p must have a remainder of 1 when divided by 3. By adding the digits of each choice, all add up to one more than a multiple of 3 except choice A, 25,366,109.

21. D The area of the n-gon is $n\left(\frac{1}{2}(3)(3)\sin\frac{2\pi}{n}\right)$. In order for the area to be 3^3 = 27, $n\left(\sin\frac{2\pi}{n}\right)=6$. Of the choices, only n = 12 works.



22. C Let n be the number of people who voted and a be the number who voted for Don up to the time when he had 45% of the vote. Then

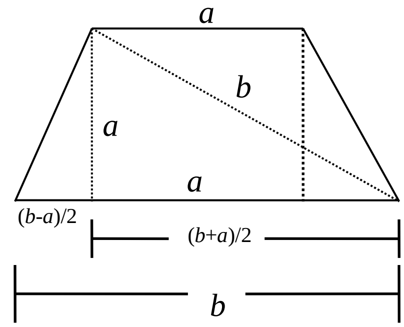
$$\frac{a}{n} = \frac{45}{100} = \frac{9}{20} \quad \Rightarrow \quad 20a = 9n.$$

From the above equation, we note that n must be even. Let x be the number of people who voted during the next five minutes, and let b be the number who voted for Don during that five-minute period. Then we have

$$\frac{a+b}{n+x} = \frac{3}{10} \quad \Rightarrow \quad 10a + 10b = 3n + 3x.$$

Since $20a + 20b = 9n + 20b$ we have $9n + 20b = 6n + 6x$ so that $3n + 20b = 6x$. We note that x is as small as possible when b is as small as possible. Clearly, the equation above has a solution if $b = 0$, which gives $n = 2x$. Thus, $20a = 18x$, or $10a = 9x$. The smallest integer solution to this equation is $x = 10$ and $a = 9$. Therefore at least 10 people voted in the five minute period.

23. B Represent the shorter base and the altitude by a, the longer base and diagonal by b, and the other segments as indicated. Using the Pythagorean Theorem:



$$a^2 + \left(\frac{b+a}{2}\right)^2 = b^2 \quad \Rightarrow \quad a^2 + \frac{b^2 + 2ab + a^2}{4} = b^2$$

Simplifying, this becomes

$5a^2 + 2ab - 3b^2 = 0 \Rightarrow$ $(5a - 3b)(a + b) = 0$. Therefore, $5a - 3b = 0$ and $\frac{a}{b} = \frac{3}{5}$.

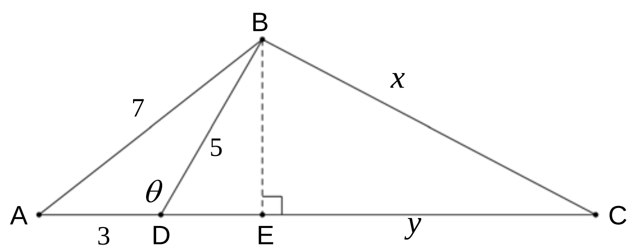
24. E Note that x must be even and z must be odd. Adding the two equations and solving for $x + z$ yields $x + z = \frac{4035 - 4y}{3} = 1345 - \frac{4y}{3}$.

Since x, y, and z are all positive integers, y must be a multiple of 3. As y takes on the values 3, 6, 9, …, the value of $x + z$ decreases by 4 each time. Therefore, the smallest value for $x + y + z$ will occur when y is as large as possible. When $y = 1008$, $x + y = 1$, which is not possible since both must be positive integers. When $y = 1005$, $x + z = 5$. Therefore, the possible combinations are $x = 4$, $z = 1$ or $x = 2$, $z = 3$. The first of these doesn't satisfy the given equations, but the second does. Therefore, x = 2, y = 1005, z = 3 and $x + y + z = 1010$.

25. D Construct the perpendicular from B to side $\overline{AC}$ at point E. Using the Law of Cosines on ΔABD,

$$7^2 = 3^2 + 5^2 - 2(3)(5)\cos\,\theta \quad \Rightarrow \quad \cos\,\theta = -\frac{1}{2}$$

Therefore, $\theta = 120°$, making ΔBDE a 30-60-90 triangle. Thus, DE = $\frac{5}{2}$ and BE = $\frac{5\sqrt{3}}{2}$.



Using the Pythagorean Theorem on ΔBEC and solving for y, we obtain $y = \frac{\sqrt{4x^2 - 75}}{2}$.

Since AC must be an integer, and DE = $\frac{5}{2}$, $4x^2 - 75$ must be the square of an integer.

Trying each of the choices,

$4(13^2) - 75 = 601$ is not a perfect square.

$4(15^2) - 75 = 825$ is not a perfect square.

$4(17^2) - 75 = 1081$ is not a perfect square.

$4(19^2) - 75 = 1369 = 37^2$.

Therefore, the only possible choice is D.