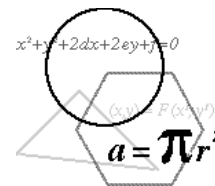


**THE 2018–2019 KENNESAW STATE UNIVERSITY  
HIGH SCHOOL MATHEMATICS COMPETITION**



**PART I – MULTIPLE CHOICE**

*For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend, or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are given if no box is marked. Zero points are given for an incorrect answer or if multiple boxes are marked. Note that wild guessing is likely to lower your score. When the exam is over, give your answer sheet to your proctor. You may keep your copy of the questions.*

**NO CALCULATORS**

**90 MINUTES**

1. Most of Harry Potter’s classmates at Hogwarts School of Witchcraft and Wizardry plan to attend the school’s annual Halloween Party. What is the minimum number of students who would have to attend in order to guarantee that there are at least 3 students born on the same day of the week and in the same month of the year.

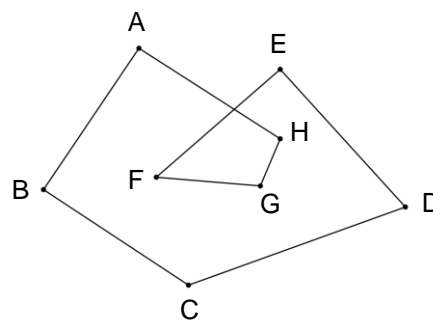


- (A) 84      (B) 144      (C) 145      (D) 168      (E) 169

2. A two-digit positive integer  $N$  has an interesting property. The sum of the digits of  $N$ ,  $N$  itself, and the number obtained by reversing the digits of  $N$  form an arithmetic sequence in that order. Compute the sum of all such integers  $N$ .

- (A) 72      (B) 96      (C) 108      (D) 120      (E) 144

3. The diagram at the right consists of eight line segments with adjoining endpoints. Compute the sum of the degree measures of angles  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ ,  $G$ , and  $H$ .



- (A) 540      (B) 630      (C) 720      (D) 810      (E) 900

4. The number of miles per hour an automobile is traveling is 21 less than the number of feet per second it is traveling. Compute the number of miles per hour in the rate of the automobile. (Note: 1 mile = 5,280 feet)

- (A) 45      (B) 48      (C) 50      (D) 52      (E) 55

5. If one of the girls at a Halloween party leaves, then 20% of the people remaining at the party are girls. If, instead, another girl arrives at the party, then 25% of the people at the party are girls. How many boys are at the party?

- (A) 16      (B) 18      (C) 20      (D) 22      (E) 24

6. The symbols  $34_r$  and  $47_s$  represent two-digit numbers in bases  $r$  and  $s$  respectively. If  $34_r = 47_s$  what is the smallest possible positive value for  $r + s$ ?
- (A) 18      (B) 19      (C) 20      (D) 21      (E) 22
7. How many integers  $x$  satisfy the equation  $(x^2 - x - 1)^{x+2} = 1$ ?
- (A) 1      (B) 2      (C) 3      (D) 4      (E) 5
8. A car travels 1 mile at 45 miles per hour. What must its average speed be, in miles per hour, for the next 2 miles in order to average 60 miles per hour for the 3 miles?
- (A) 68      (B) 70      (C) 72      (D) 75      (E) None of these
9. One of the roots of the equation  $x^3 - 21x^2 + 26x + c = 0$  is the average of the other two roots. Compute the value of  $c$ .
- (A) 546      (B) 504      (C) 486      (D) 442      (E) 404
10. Let  $p$  and  $q$  be two prime numbers for which  $\frac{36}{p} + \frac{36}{q} = \left(\frac{36}{p}\right)\left(\frac{36}{q}\right)$ , with  $p < q$ . Compute the largest possible value of the product  $pq$ .
- (A) 155      (B) 203      (C) 299      (D) 323      (E) 345
11. The sides of a triangle have lengths  $n - 1$ ,  $n$ , and  $n + 1$ , and its area is  $n\sqrt{n}$ . What is the value of  $n$ ?
- (A) 5      (B) 5.5      (C) 6      (D) 6.5      (E) 7
12. The value of  $x$  exceeds  $y$  by the square root of  $y$ . The value of  $y$  exceeds  $z$  by the square root of  $z$ . If the value of  $x$  is one greater than the value of  $z$ , compute the sum  $x + y + z$ .
- (A)  $\frac{5}{3}$       (B)  $\frac{6}{5}$       (C)  $\frac{7}{4}$       (D)  $\frac{7}{5}$       (E)  $\frac{8}{3}$
13. Which of the following values of  $n$  satisfies the equation  $(1 + i)^n = n^2$ , where  $i$  is the imaginary unit?
- (A) 4      (B) 8      (C) 12      (D) 16      (E) 20
14. Given the pair of equations  $2x - 3y = 13$  and  $3x + 2y = b$ , where  $b$  is an integer and  $1 \leq b \leq 200$ . For how many values of  $b$  will  $x + y$  be the square of an integer?
- (A) 1      (B) 2      (C) 3      (D) 4      (E) 5

15. The line  $y = mx + 1$  ( $m > 0$ ) intersects the ellipse  $x^2 + 4y^2 = 1$  in exactly one point. If  $\theta$  is the acute angle this line makes with the  $x$ -axis, compute  $\sin^2 \theta$ .

(A)  $\frac{1}{3}$       (B)  $\frac{1}{4}$       (C)  $\frac{1}{7}$       (D)  $\frac{3}{4}$       (E)  $\frac{3}{7}$

16. Debbie and Don were comparing their stacks of pennies. Debbie said “If you gave me a certain number of pennies from your stack, then I’d have six times as many as you, but if I gave you that number, you’d have one-third as many as me.” What is the smallest number of pennies that Debbie could have had?

(A) 28      (B) 35      (C) 40      (D) 42      (E) 45

17. The parallel sides of a trapezoid are 12 and 17 inches long, and the nonparallel sides are 3 and 4 inches long. Compute the area of the trapezoid.

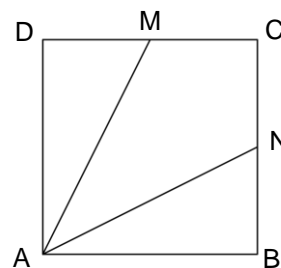
(A) 32.4      (B) 34.8      (C) 43.5      (D) 48.5      (E) None of these

18. A palindrome is a number that reads the same forwards and backwards. For example, 23732 is a palindrome. The digits 1, 1, 1, 1, 2, 2, 3, 3, and 4 are all to be used in writing a nine-digit number. If the digits are randomly placed in a row, what is the probability that the number will be a palindrome?

(A)  $\frac{1}{315}$       (B)  $\frac{1}{275}$       (C)  $\frac{1}{225}$       (D)  $\frac{1}{195}$       (E) None of these

19. In the figure, M and N are midpoints of two adjacent sides of square ABCD. Compute  $\tan(\angle MAN)$ .

(A)  $\frac{1}{2}$       (B)  $\frac{2}{3}$       (C)  $\frac{3}{4}$       (D)  $\frac{\sqrt{3}}{3}$       (E)  $\sqrt{3}$



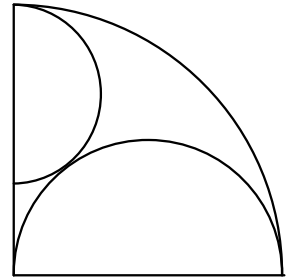
20. The points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  lie on the graph  $y = \log_a x$ , and  $x_2 > x_1$ . The horizontal line through the midpoint of segment PQ intersects the graph of  $y = \log_a x$  at the point  $N(x_3, y_3)$ . Which of the following is an equivalent expression for  $x_3$ ?

(A)  $\frac{1}{2}(x_1 + x_2)$       (B)  $\frac{1}{2}\log_a(x_1 + x_2)$       (C)  $\frac{1}{2}(\log_a x_1 + \log_a x_2)$   
 (D)  $\sqrt{(x_1)(x_2)}$       (E)  $\sqrt{\frac{x_2}{x_1}}$

21. In Dr. Garner's math class, the number of students receiving A's is at least one-fifth of the number of students receiving B's and at most one-sixth the number of students receiving C's. The number of students receiving A's or B's is at least 23. If A, B, and C are the only grades given in the class, what is the minimum number of students receiving C's?

(A) 18      (B) 20      (C) 22      (D) 23      (E) 24

22. In the diagram, a semicircle is inscribed in a quarter circle. A second semicircle, tangent to the first, is inscribed in the quarter circle, as shown. Compute the ratio of the diameter of the smaller semicircle to the diameter of the larger semicircle.



(A)  $\frac{2}{3}$       (B)  $\frac{3}{5}$       (C)  $\frac{5}{8}$       (D)  $\frac{7}{10}$       (E) None of these

23. Let  $x$  and  $y$  be two positive integers such that  $x^2 + y$  is exactly 2018 more than  $x + y^2$ . Compute the sum of all possible values of  $x$ .

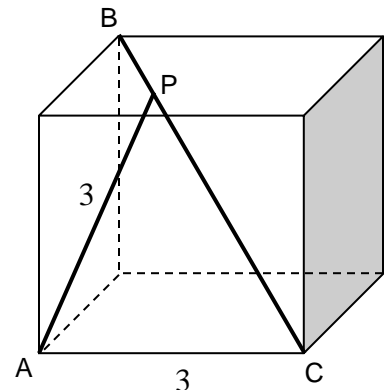
(A) 1472      (B) 1516      (C) 1604      (D) 1648      (E) 1696

24. Let  $\lfloor x \rfloor$  represent the greatest integer less than or equal to  $x$ . For example,  $\lfloor \pi \rfloor = 3$  and  $\lfloor 8 \rfloor = 8$ . How many positive integers  $n$ , with  $n \leq 2018$ , yield a solution for  $x$  (where  $x$  is real) in the equation  $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n$ ?

(A) 1346      (B) 1598      (C) 1604      (D) 2018      (E) None of these

25. The diagram shows a cube whose edge is 3 inches long. A line segment is drawn from vertex A of the cube to point P on the main diagonal,  $\overline{BC}$ , so that the length of segment  $\overline{AP}$  is also 3 inches.

Compute the ratio  $\frac{BP}{PC}$ .



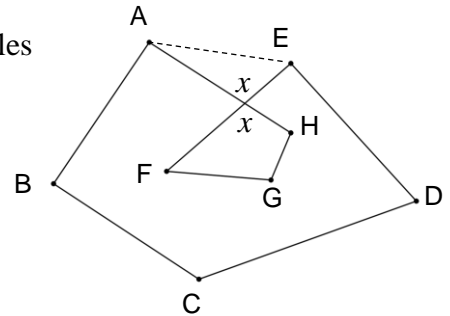
(A)  $\frac{1}{2}$       (B)  $\frac{1}{3}$       (C)  $\frac{1}{4}$       (D)  $\frac{2}{5}$       (E)  $\frac{3}{8}$

## Solutions

1. **E** The greatest number of people a group can have with no two born in the same month and on the same day of the week is  $(12)(7) = 84$ . The greatest number of people a group can have with no three born in the same month and on the same day of the week is  $2(84) = 168$ . Add one more person, and there must be at least 3 born in the same month and on the same day. Therefore, the answer is 169.

2. **D** Let  $N = 10t + u$ . Then the arithmetic sequence is  $t + u, 10t + u, 10u + t$ . Then  
 $10t + u - (t + u) = 10u + t - (10t + u) \Rightarrow 9t = 9u - 9t \Rightarrow u = 2t$ .  
 Therefore, the only possible values of  $N$  are 12, 24, 36, and 48, with a sum of 120.

3. **C** Construct segment  $AE$ . The sum of the measures of the angles of pentagon  $ABCDE$  is  $180(5 - 2) = 540$ . The sum of the measures of angles  $F, G,$  and  $H$  is  $360 - x$ . The sum of the measures of  $\angle HAE$  and  $\angle FEA$  is  $180 - x$ . The desired sum is  $540 - (180 - x) + (360 - x) = 720$



4. **A** 1 mile per hour = 88 feet per minute =  $\frac{22}{15}$  feet per second. Therefore,  $x$  mph =  $\frac{22}{15}x$  feet per second. We must solve  $x = \frac{22x}{15} - 21 \Rightarrow 7x = 315$  and  $x = 45$ .

5. **E** Let  $w$  be the number of girls at the party and  $m$  be the number of boys.  
 Then  $\frac{w-1}{w+m-1} = \frac{1}{5} \Rightarrow 4w - m = 4$  and  $\frac{w+1}{w+m+1} = \frac{1}{4} \Rightarrow 3w - m = -3$ .  
 Solving the two equations,  $w = 7$  and  $m = 24$ . Therefore, there were 24 boys at the party.

6. **E** First note that  $r > 4$  and  $s > 7$ . Since  $34_r = 3r + 4$  and  $47_s = 4s + 7$ , we get  
 $3r + 4 = 4s + 7$  and  $3r = 4s + 3$ . The first value of  $s > 7$  for which this is true is  $s = 9$  and  
 $r = 13$ . Therefore, the desired sum is 22.

7. **D** There are three ways in which  $a^b$  can equal 1 when  $a$  and  $b$  are integers:  
 (i)  $a = 1$ , (ii)  $a = -1$  and  $b$  is even, or (iii)  $b = 0$  and  $a \neq 0$ .  
 (i)  $x^2 - x - 1 = 1 \Rightarrow x = 2$  or  $-1$ .  
 (ii)  $x^2 - x - 1 = -1$  and  $x + 2$  is even.  
 $x^2 - x - 1 = -1 \Rightarrow x = 0$  or  $1$ . If  $x = 0$ ,  $x + 2$  is even, but if  $x = 1$ ,  $x + 2$  is odd.  
 (iii)  $x + 2 = 0$  and  $x^2 - x - 1 \neq 0$ . When  $x = -2$ ,  $x^2 - x - 1 = 5 \neq 0$ .  
 Therefore, there are a total of 4 solutions (namely,  $-2, -1, 0,$  and  $2$ ).

8. **C** Let  $x$  be the speed for the next two miles. Average speed =  $\frac{\text{total distance}}{\text{total time}}$ . Therefore,  
 $\frac{3}{1/45 + 2/x} = 60$  and solving,  $x = 72$ .

9. **B** In any equation of the form  $x^3 + ax^2 + bx + c = 0$  with roots  $p, q,$  and  $r,$   
 (i)  $p + q + r = -a,$  (ii)  $pq + pr + qr = b,$  and (iii)  $pqr = -c$

The three roots of the given equation  $x^3 - 21x^2 + 26x + c = 0$  may be represented by  $p - d, p,$  and  $p + d.$

Using (i)  $(p - d) + p + (p + d) = 21$  and  $p = 7.$

The value of  $c$  can be found by substituting in the equation, or...

using (ii)  $7(7 - d) + 7(7 + d) + (7 - d)(7 + d) = 26$  and  $d = \pm 11.$

Therefore, the three roots are  $-4, 7,$  and  $18.$  Using (iii)  $c = -(-4)(7)(18) = 504.$

10. **D**  $\frac{36}{p} + \frac{36}{q} = \left(\frac{36}{p}\right)\left(\frac{36}{q}\right) \Rightarrow 36q + 36p = 36^2 \Rightarrow p + q = 36.$  The combinations of two primes  $(p, q)$  that satisfy this equation with  $p < q$  are  $(5, 31), (7, 29), (13, 23),$  and  $(17, 19).$  Of these the largest product  $pq$  is  $(17)(19) = 323.$

11. **C** The semiperimeter of the triangle is  $\frac{3n}{2}.$  Using Heron's Formula,

$$n\sqrt{n} = \sqrt{\frac{3n}{2}\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} - 1\right)\frac{n}{2}}$$

Simplifying,  $3n^2 - 16n - 12 = 0$  giving  $(3n + 2)(n - 6) = 0,$  so that  $n = 6.$

12. **A**  $x = y + \sqrt{y}, y = z + \sqrt{z},$  and  $x = z + 1$

$$z + 1 = y + \sqrt{y} = z + \sqrt{z} + \sqrt{y} \Rightarrow 1 = \sqrt{z} + \sqrt{z + \sqrt{z}} \Rightarrow 1 - \sqrt{z} = \sqrt{z + \sqrt{z}}.$$

Squaring both sides of the last equation,  $1 - 2\sqrt{z} + z = z + \sqrt{z}$  or  $3\sqrt{z} = 1$  and  $z = \frac{1}{9}.$

Therefore,  $x = z + 1 = \frac{10}{9},$  and  $y = z + \sqrt{z} = \frac{4}{9}.$  Hence,  $x + y + z = \frac{10}{9} + \frac{4}{9} + \frac{1}{9} = \frac{5}{3}.$

13. **D** Look for a pattern:

$$(1 + i)^2 = 2i$$

$$(1 + i)^4 = (2i)^2 = -4$$

$$(1 + i)^8 = (-4)^2 = 16$$

$$(1 + i)^{16} = 16^2$$

Therefore,  $n = 16$

14. **D** Eliminate  $y$  from the given equations to obtain  $13x = 26 + 3b$  and  $x = 2 + \frac{3b}{13}.$

Eliminating  $x$  from the given equations and solving for  $y,$  we obtain  $y = -3 + \frac{2b}{13}.$

Therefore,  $x + y = \frac{5b}{13} - 1.$  In order for  $x + y$  to be an integer,  $b$  must be a multiple

of 13. The only multiples of 13 less than or equal to 200 for which  $x + y$  is a square are:

$b = 13, x + y = 4 = 2^2;$   $b = 26, x + y = 9 = 3^2;$   $b = 130, x + y = 49 = 7^2;$  and

$b = 169, x + y = 64 = 8^2.$  Thus, there are 4.

15. **E** Substituting  $y = mx + 1$  into  $x^2 + 4y^2 = 1$  and rearranging terms, we obtain

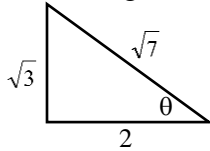
$$(4m^2 + 1)x^2 + 8mx + 3 = 0$$

In order for the two graphs to intersect exactly once, the discriminant must be zero.

$$(8m)^2 - 4(4m^2 + 1)(3) = 16m^2 - 12 = 0 \text{ and } m = \frac{\sqrt{3}}{2}.$$

Since the slope of a line is the tangent of the angle the line makes with the positive  $x$ -axis,

$$\tan \theta = \frac{\sqrt{3}}{2} \text{ from which } \sin^2 \theta = \frac{3}{7}.$$



16. **E** Let  $a$  and  $b$  be, respectively, the number of pennies that Debbie and Don had in their stacks, and let  $x$  be the certain number of pennies. From the given information, we obtain the following two equations:  $a + x = 6(b - x)$  and  $a - x = 3(b + x)$ . From the first equation,  $a = 6b - 7x$ , and, from the second equation,  $a = 3b + 4x$ . Therefore,  $6b - 7x = 3b + 4x$ , from which  $b = \frac{11}{3}x$ . Since  $a$ ,  $b$ , and  $x$  are required to be positive integers, the smallest possible value for  $x$  is 3. Then  $b = 11$  and  $a = 6(11) - 7(3) = 45$ . Therefore, 45 is the smallest number of pennies that Debbie could have had.

17. **B** Method 1:

In the diagram, construct a line through D parallel to leg BC. Quadrilateral DEBC is a parallelogram, making  $\triangle ADE$  a 3-4-5 right triangle. The area

of  $\triangle ADE$  is  $\frac{1}{2}(3)(4) = 6$ . Thus, the altitude

from D to AE has length  $\frac{12}{5} = 2.4$ . Therefore, the area of trapezoid ABCD is

$$\frac{1}{2}(2.4)(12 + 17) = 34.8$$

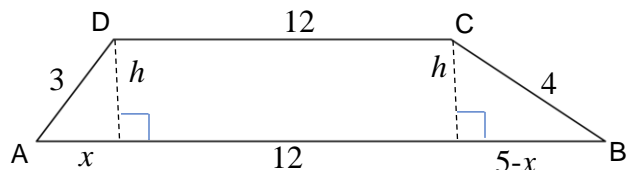
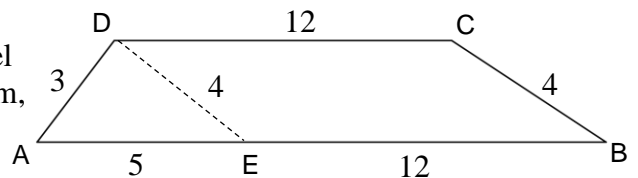
Method 2:

In the diagram, draw altitudes from D and C and represent the lengths of the segments

as shown. Then,  $h^2 = 9 - x^2 = 16 - (5 - x)^2$

from which  $x = \frac{9}{5} = 1.8$ . Thus,  $h = \sqrt{9 - (1.8)^2} = 2.4$ , and the area of trapezoid ABCD

is  $\frac{1}{2}(2.4)(12 + 17) = 34.8$



18. **A** The number of distinct arrangements of the nine digits is  $\frac{9!}{(4!)(2!)(2!)} = 3780$ . A palindrome will have 4 in the fifth position and two 1's and one each of the 2's and 3's in positions 1 – 4. Positions 6 – 9 will mirror 1 – 4. The number of distinct ways of arranging these numbers is  $\frac{4!}{2!} = 12$ . The probability that the number is a palindrome is

$$\frac{12}{3780} = \frac{1}{315}.$$

19. **C** Method 1:

Since  $\triangle DMA \cong \triangle BNA$ ,  $\angle DAM \cong \angle BAN$ . Represent the measure of each as  $\alpha$ . Without loss of generality, let the length of the sides of the square be 2. Then  $DM = NB = 1$

and  $AN = AM = \sqrt{5}$ , and  $\cos \alpha = \frac{2}{\sqrt{5}}$ .

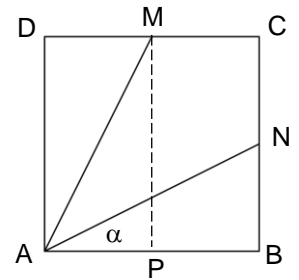
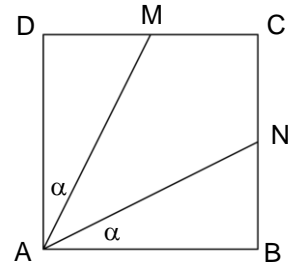
$$\sin(\angle MAN) = \sin(90 - 2\alpha) = \cos 2\alpha = 2\cos^2\alpha - 1 = 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 = \frac{3}{5}.$$

$$\text{Thus } \cos(\angle MAN) = \frac{4}{5} \text{ and } \tan(\angle MAN) = \frac{3}{4}.$$

Method 2:

Let  $m(\angle MAB) = \beta$ . Construct  $\overline{MP} \perp \overline{AB}$ . Then  $\tan \alpha = \frac{1}{2}$  and  $\tan \beta = 2$ .

$$\text{Since } m(\angle MAN) = \beta - \alpha, \tan(\angle MAN) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)} = \frac{3}{4}$$



20. **D** The coordinates of the midpoint of segment PQ are the average of the coordinates of P and Q,  $\left(\frac{x_1 + x_2}{2}, \frac{\log_a x_1 + \log_a x_2}{2}\right)$ . Since the line through the midpoint of segment PQ and point N is horizontal, the y-coordinate of point N is the same as the y-coordinate of the midpoint. Therefore,  $y_3 = \log_a x_3 = \frac{\log_a x_1 + \log_a x_2}{2}$ .

Using the laws of logarithms,  $\log_a x_3 = \log_a [(x_1)(x_2)]^{\frac{1}{2}}$ , from which  $x_3 = \sqrt{(x_1)(x_2)}$ .

21. **E** Let A, B, and C represent the number of students receiving each of the three grades. Then

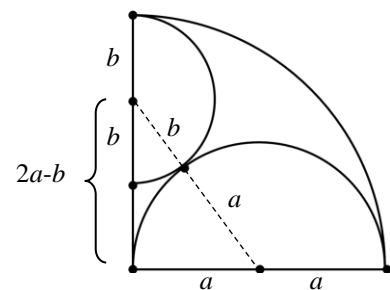
$$\frac{1}{6}C \geq A \geq \frac{1}{5}B \text{ and } A + B \geq 23. \text{ Since } A \geq \frac{1}{5}B, 5A \geq B, \text{ and } 6A \geq B + A \geq 23, \text{ then}$$

$$6A \geq 23 \text{ and } A \geq \frac{23}{6}. \text{ But } A \text{ must be a positive integer, so } A \geq 4. \text{ Since } \frac{1}{6}C \geq A \geq 4,$$

we have  $\frac{1}{6}C \geq 4$  and  $C \geq 24$ . The values  $C = 24$ ,  $A = 4$ , and  $B = 20$  are consistent with the given information. Therefore, the minimum number of students receiving C's is 24.



22. **A** Represent the diameter of the larger semicircle as  $2a$ , and the diameter of the smaller semicircle as  $2b$ . Note that the line joining the centers of the two semicircles passes through their common point. Using the Pythagorean Theorem,
- $$(2a - b)^2 + a^2 = (a + b)^2 \Rightarrow 4a^2 - 4ab + b^2 + a^2 = a^2 + 2ab + b^2.$$
- Simplifying,  $4a^2 = 6ab \Rightarrow \frac{2b}{2a} = \frac{2}{3}.$



23. **B** We are looking for the solutions to the equation  $(x^2 + y) - (x + y^2) = 2018$ .  
 Rewriting the left side as  $(x^2 - y^2) - (x - y)$  and factoring, the equation becomes  $(x - y)(x + y - 1) = 2018$ . Since  $x \geq 1$  and  $y \geq 1$ , then  $x + y - 1$  is positive, and thus  $x - y$  is positive. Therefore,  $x - y$  and  $x + y - 1$  are positive integral factors of 2018, with the first smaller than the second. There are two possibilities. Either  $x - y = 1$  and  $x + y - 1 = 2018$ , or  $x - y = 2$  and  $x + y - 1 = 1009$  (note that 1009 is prime). In the first case,  $x = 1010$  and  $y = 1009$ . In the second case,  $x = 506$  and  $y = 504$ . The desired sum is 1516.

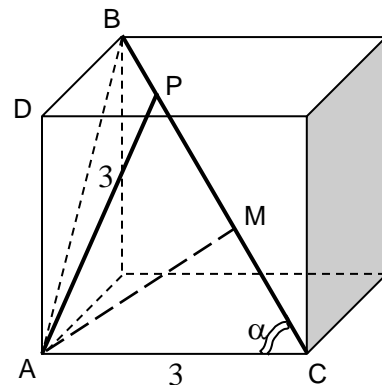
24. **A** Let  $k$  be a non-negative integer. Let  $f(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$ . If  $k \leq x < k + \frac{1}{3}$ , then  $f(x) = 6k$ . If  $k + \frac{1}{3} \leq x < k + \frac{1}{2}$ , then  $f(x) = 6k + 1$ . If  $k + \frac{1}{2} \leq x < k + \frac{2}{3}$ , then  $f(x) = 6k + 2$ . If  $k + \frac{2}{3} \leq x < k + 1$ , then  $f(x) = 6k + 3$ . Therefore, there is only a solution if  $n$  is congruent to 0, 1, 2, or 3 mod 6. Thus, there are  $(2016)\frac{4}{6} + 2 = 1346$  such positive integers  $n$ .

25. **A** Method 1:

Draw segment  $AB$ . Using the Pythagorean Theorem on isosceles right triangle  $ABD$ ,  $AB = 3\sqrt{2}$ . Since segments  $AB$  and  $AC$  are perpendicular,  $\triangle ABC$  a right triangle, and  $BC = 3\sqrt{3}$ . Let  $\angle PCA = \alpha$ . Then  $\cos \alpha = \frac{AC}{BC} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$ .

Construct the altitude  $AM$  to side  $PC$  of isosceles  $\triangle APC$ . Then in right triangle  $AMC$ ,  $\cos \alpha = \frac{1}{\sqrt{3}} = \frac{MC}{AC} = \frac{MC}{3}$ , and  $MC = \frac{3}{\sqrt{3}} = \frac{1}{2}(PC)$ .

Thus  $PC = 2\left(\frac{3}{\sqrt{3}}\right) = 2\sqrt{3}$ ,  $BP = \sqrt{3}$ , and the desired ratio  $\frac{BP}{PC} = \frac{1}{2}$ .



Method 2:

Draw segment AB. As in method 1,  $AB = 3\sqrt{2}$ ,  $BC = 3\sqrt{3}$ , and  $\cos \alpha = \frac{1}{\sqrt{3}}$ . Since  $\triangle APC$  is isosceles,  $\angle APC = \alpha$ . Then,

$$\cos \theta = \cos(180 - 2\alpha) = -\cos 2\alpha = -(2\cos^2 \alpha - 1) = 1 - 2\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}.$$

Using the Law of Cosines in  $\triangle APC$ ,

$$PC^2 = AC^2 + AP^2 - 2(AC)(AP)\cos \theta = 3^2 + 3^2 - 2(3)(3)\left(\frac{1}{3}\right) = 12$$

and  $PC = 2\sqrt{3}$ . Thus,  $BP = \sqrt{3}$ , and the desired ratio  $\frac{BP}{PC} = \frac{1}{2}$ .

