KENNESAW STATE UNIVERSITY
COLLEGE OF SCIENCE AND MATHEMATICS
Department of Mathematics

THE 2022–2023 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION

PART II

Calculators are NOT permitted
Time allowed: 2 hours

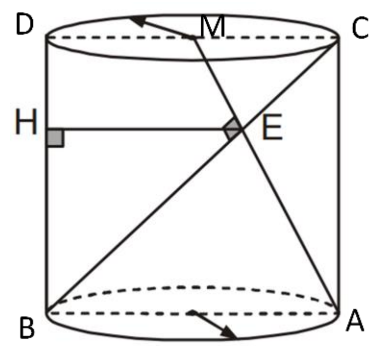
1. If $\boldsymbol{a}$ is an even positive integer and $A = \boldsymbol{a}^n + \boldsymbol{a}^{n-1} + \cdots + \boldsymbol{a} + 1$ is a perfect square, prove that $\boldsymbol{a}$ is a multiple of 8.

2. Prove $2022^{2023} + 2023^{2022} \neq n^2$ for any integer n.

3. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$ Find the minimum value for:

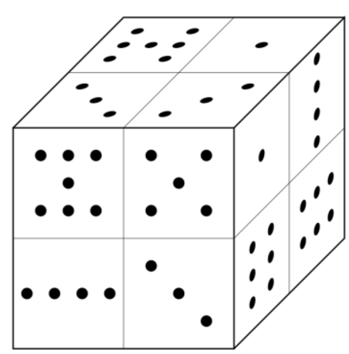
$$K = \frac{(cosx + secx)^6 - \cos^6 x - \sec^6 x - 2}{(cosx + secx)^3 + \cos^3 x + \sec^3 x}$$

4. In the diagram, we have a right circular cylinder, where; $\overline{HE}$ is perpendicular to $\overline{BD}, \overline{HE} = 4$, $\overline{AM}$ is perpendicular to $\overline{BC}$ and the intersection point is E, and M is the center of the upper base. Find with proof the total surface area of this right cylinder. Please include a diagram with your proof.



5. Eight identical cubes with a variety of dots on their faces were glued together to form a larger cube shown on the graph.

   What is the smallest total number of dots that could be on the three hidden faces of the big cube? Justify your answer and include appropriate diagrams. It should be noted that the eight small cubes are not traditional dice.



Solutions

1. Since a is even $\Rightarrow$ A is odd $\Rightarrow$ $A = (2k+1)^2 = 4k(k+1)+1 = 8m+1$ (because k or $k+1$ is even).
   $\Rightarrow$ $a^n + a^{n-1} + \cdots + a = A - 1 = 8m$
   $\Rightarrow$ $a(a^{n-1} + a^{n-2} + \cdots + 1) = 8m$
   $\Rightarrow$ 8 divides a because $a^{n-1} + a^{n-2} + \cdots + 1$ is odd.

2. The idea here is to look at the units-digit of 2022^{2023} + 2023^{2022} and the units digit of any square. Any positive integer can be written as $10k$ + u, where k is an integer [For example: 2023 = 10(202) + 3]. The units-digit of $(10k + u)^n$ ends in the same digit as u^n since the product of any number and any power of 10k ends in zero and has no effect on the units-digit of $(10k + u)^n$. Therefore, 2022^{2023} has the same units-digit as 2^{2023} = $(2^4)^{505}(2^3) = 16^{505}(8)$. Since any power of 6 ends in a 6, the units-digit of 2022^{2023} is 8.

   Similarly, 2023^{2022} ends in the same units-digit as 3^{2022} = $(3^2)^{1011} = (9)^{1011} = (9)^{1010}(9)$ = $(9^2)^{505}(9) = (81)^{505}(9)$, and any power of 1 ends in a 1. Therefore, 2022^{2023} + 2023^{2022} ends in a units-digit of 8 + 9 = …7. Since no perfect square ends in a units-digit of 7, 2022^{2023} + 2023^{2022} $\neq n^2$ for any integer n.

   Alternate solution. We can try to prove that a number is not a perfect square by taking it modulo m, for some small number m, and seeing that no perfect square has that remainder modulo m. But which value of m to pick?

   In this case, we can try to make our calculations easier by picking a divisor of 2023. The prime factorization of 2023 is $7 \cdot 17 \cdot 17$, so let's try taking $2022^{2023} + 2023^{2022}$ modulo 7. We have $2022 \equiv$ -1 (mod 7), so $2022^{2023} \equiv$ $(-1)^{2023} \equiv$ -1 $\equiv 6$ (mod 7). Meanwhile, $2023^{2022} \equiv 0$ (mod 7) because 2023 is divisible by 7. Therefore their sum is congruent to 6 modulo 7.

   Meanwhile, we can check that $0^2 \equiv 0$ (mod 7), $1^2 \equiv 1$ (mod 7), $2^2 \equiv 4$ (mod 7), $3^2 \equiv 2$ (mod 7), $4^2 \equiv 2$ (mod 7), $5^2 \equiv 4$ (mod 7), and $6^2 \equiv 1$ (mod 7). Therefore n^2 leaves a remainder of 0, 1, 2, or 4 modulo 7: never 6. It follows that $2022^{2023} + 2023^{2022}$ can never be equal to n^2 for any integer n.

   Alternate solution. By observing that 2023 = 7 * 17^2 and consider everything (mod 7). One can see that 2022^2023 = (-1)^2023 = -1 (mod 7) while 2023^2022 = 0 (mod 7). Hence, 2022^2023 + 2023^2022 = -1 = 6 (mod 7). However, all perfect squares are = 0, 1, 2 or 4 (mod 7) which makes 2022^2023 + 2023^2022 impossible to be a perfect square.

3. **Lemma**: For any two positive numbers x and y we have $x + y \geq 2\sqrt{xy}$.
   Proof: This follows from $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$.

   Let $a = cosx$, $b = secx \ \Rightarrow ab = 1$

   $$\Rightarrow \ K = \frac{(a+b)^6 - a^6 - b^6 - 2}{(a+b)^3 + (a^3+b^3)} = \frac{(a+b)^6 - (a^3+b^3)^2}{(a+b)^3 + (a^3+b^3)}$$

   $$\Rightarrow \ K = \frac{\left[(a+b)^3 + (a^3+b^3)\right]\left[(a+b)^3 - \left(a^3+b^3\right)\right]}{(a+b)^3 + (a^3+b^3)} \qquad \Rightarrow \ K = (a+b)^3 - (a^3+b^3)$$

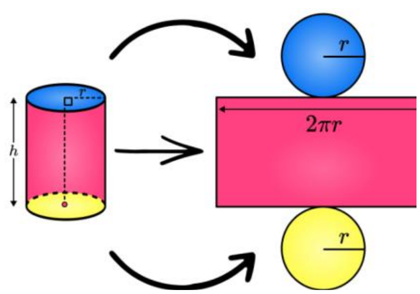
   $$\Rightarrow \ K = 3ab(a+b) \qquad \Rightarrow \ K = 3(cosx \ + secx)$$

   Because $-\frac{\pi}{2} < x < \frac{\pi}{2} \ \Rightarrow \ \ cosx > 0$ and $secx > 0$. So, applying lemma

   $cosx \ + secx \geq 2 \ \Rightarrow \ \ K \geq 6$

4. Surface area of a right cylinder.

   $$S = 2\pi r^2 + 2\pi rh$$

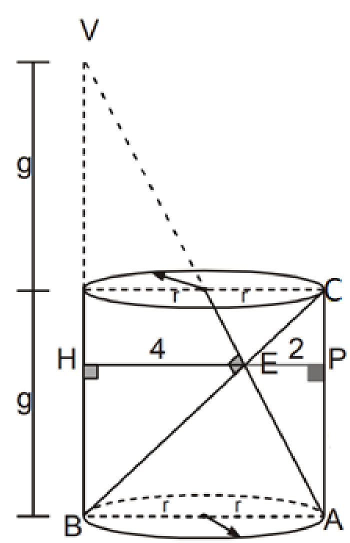


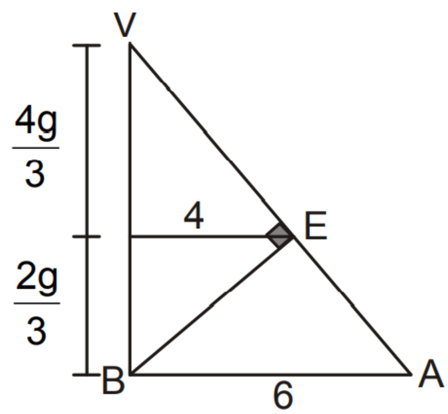
   Prolonging BH and AE to V, we get similar triangles ΔABV and ΔAEC:

   $$\Rightarrow \frac{2g}{4} = \frac{g}{EP}$$

   $$\Rightarrow \overline{EP} = 2$$

   $$\Rightarrow \overline{HP} = 6 = 2r \ \ \Rightarrow \ \ \ r = 3$$





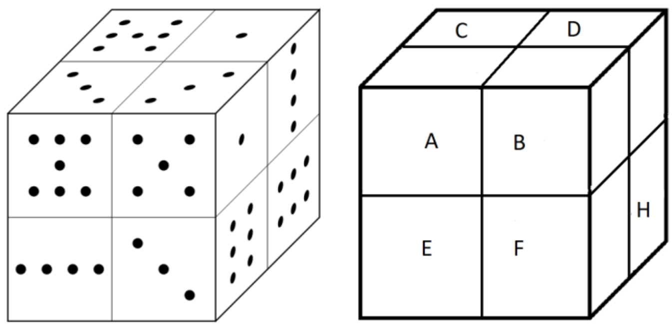
Notice on triangle ΔABV:
triangle ΔBEH is similar to ΔHEV, then

$$\frac{\frac{4g}{3}}{4} = \frac{4}{\frac{2g}{3}} \quad \Rightarrow \; g = 3\sqrt{2}$$

So,

$$S = 2\pi(3)^2 + 2\pi(3)\left(3\sqrt{2}\right) = 18\pi + 18\sqrt{2}\pi$$

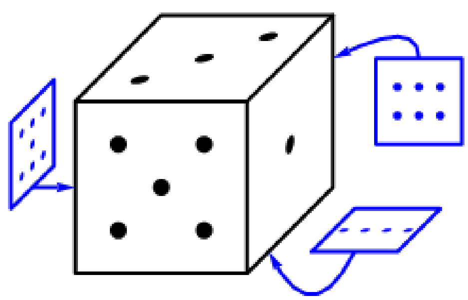
5. Let’s label the small cubes (A, B, C, D, E, F, G, H), where G is the cube completely hidden at the bottom left back:



We can observe:

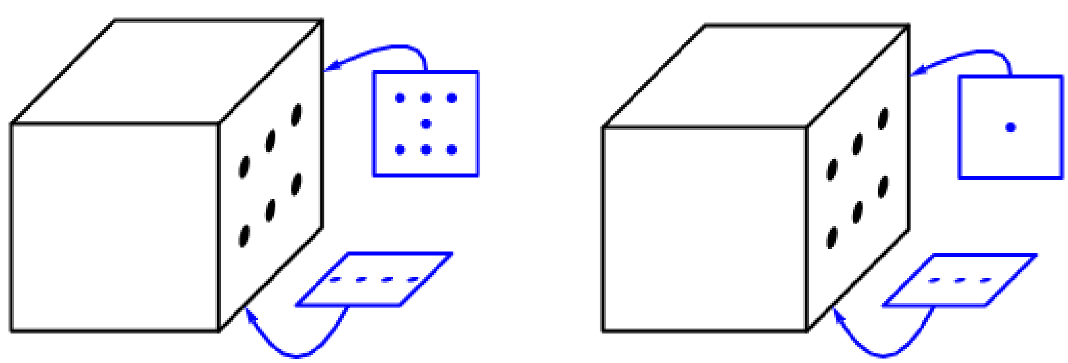
a) From cube A we can conclude that on the left face of cube B must be 7 dots.
b) Cube F has faces with 3 and 6 dots adjacent to each other, then B must have 6 dots in the faces opposite to the one with 5 dots.
c) So then (because there are 1, 3, 4, 5, 6 or 7 dots in each face) the bottom face in cube B has 4 dots.

Summary for **cube B**:



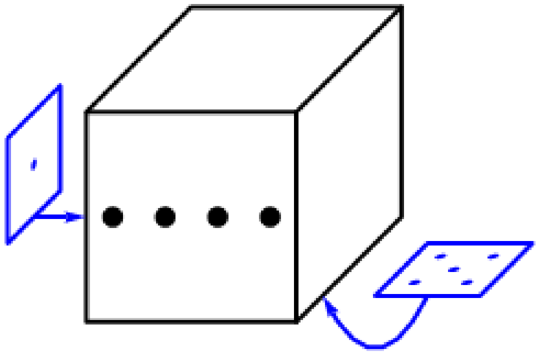
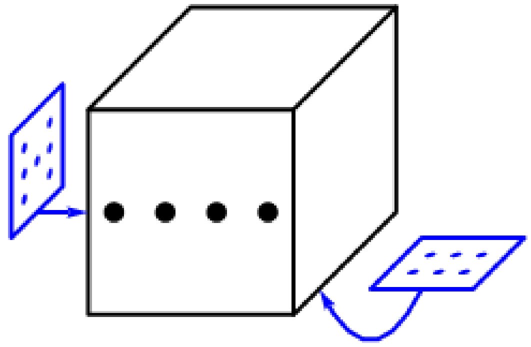
Also, from previous analysis we can conclude:

a) Cube A has 6 dots on the left face.
b) Cube F has 1 dot on the bottom face.
c) Cube D has 6 dots on the hidden face.
d) **Cube H** has two possibilities:



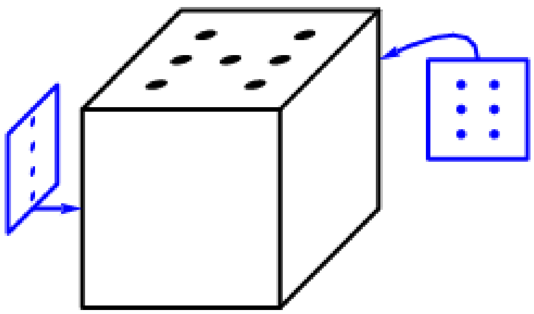
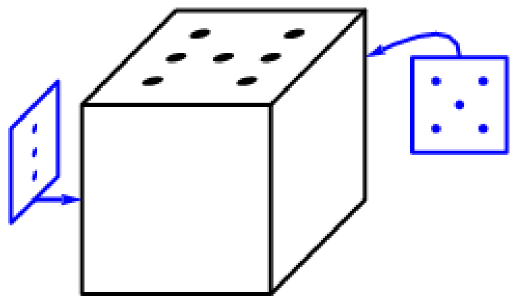
So, the minimum number of dots on the hidden face for cube H is 4.

e) **Cube E** has also two possibilities:

So, the minimum number of dots on the hidden face for cube E is 6.

f) **Cube C** has two possibilities:

So, the minimum number of dots on the hidden face for cube C is 8.

g) **Cube G** has its three faces hidden, so the minimum number of points is 1+3+5=9.

Conclusion, the minimum number of dots in the hidden faces are: 6+1+6+4+6+8+9=40.